PACIFIC JOURNAL OF MATHEMATICS
Vol. 49, No. 1, 1973

WHEN ARE WITT RINGS GROUP RINGS?

ROGER WARE

It has been shown that if C is a commutative connected semi-local ring with involution J then the Witt ring, $W(C, J)$, of hermitian forms over C is a factor ring of an integral group ring $Z[G]$, with G a group of exponent two. The purpose of this note is to characterize those pairs (C, J) whose Witt rings are actually isomorphic to integral group rings (Theorem 1).

I would like to express my thanks to Alex Rosenberg and Manfred Knebusch for several helpful suggestions.

This paper is in part motivated by the result of Elman and Lam which states that if F is a superpythagorean field [3, Th. 4.3, Def. 4.4] then the Witt ring, $W(F)$, of F is isomorphic to a group ring $\boldsymbol{Z}[H]$, where H can be taken to be any subgroup of F^*/F^{*2} of index two, not containing the square class of -1 [3, Th. 5.13 (8)]. In Theorem 1 a different proof of the Elman-Lam result is given and it is shown that the converse is also true. In order to extend the notion of superpythagorean to semi-local rings, we employ the concept of signature as defined in [6].

In what follows C will always be a commutative connected (= no idempotents other than 0 and 1) semi-local ring with involution J and A will be the fixed ring of J. We allow the possibility that J is the identity. The groups of units of C and A are denoted by C^* and A^* respectively, and $N: C^* \to A^*$ is the homomorphism given by $N(c) = cJ(c)$. We denote by $W(C, J)$ the Witt ring of hermitian spaces over C with respect to the involution J, as defined in [5]. The ring theoretic operations of $W(C, J)$ are induced by the orthogonal direct sum and tensor product of spaces respectively. For a in A^* we let $\langle a \rangle$ denote the class in $W(C, J)$ of the rank one hermitian space C with form $(c_1, c_2) \to c_1 J(c_2) a$ and $[a]$ the image of a in the group A^*/NC^*. Then $\langle a \rangle = \langle b \rangle$ in $W(C, J)$ if and only if $[a] = [b]$ in A^*/NC^* and $\langle a \rangle \langle b \rangle = \langle ab \rangle$. Hence the assignment $[a] \to \langle a \rangle$ induces a ring homomorphism $\psi: \boldsymbol{Z}[A^*/NC^*] \to W(C, J)$. By [5, Th. 1.16], the mapping ψ is surjective.

A signature σ of (C, J) is a group homomorphism $\sigma: A^* \to \{\pm 1\}$ with the property that $\sigma(NC^*) = 1$ and if $\sigma: \boldsymbol{Z}[A^*/NC^*] \to \boldsymbol{Z}$ also denotes the induced ring homomorphism then $\sigma(\operatorname{Ker} \psi) = 0$. As remarked in [6], the signatures of (C, J) correspond bijectively with the ring homomorphisms from $W(C, J)$ to $\boldsymbol{Z}$. By [5, Example 3.11] the latter set is in bijective correspondence with the set of non-maximal prime ideals of $W(C, J)$. If J is the identity and $C = A$ is a field