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BOUNDS FOR PRODUCTS OF INTERVAL FUNCTIONS

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Since it is possible for ${}_a\Pi^b(1+G)$ to exist and not be zero when G is unbounded and $1+G$ is not bounded away from zero, the conditions under which products of the form $|\Pi_1^n[1+G(x_{q-1}, x_q)]|$ are bounded or bounded away from zero for suitable subdivisions $\{x_q\}_0^n$ of $[a, b]$ are important in many theorems concerning product integrals. Conditions are obtained for such bounds to exist for products of the form $\Pi(1+FG)$ and $\Pi(1+F+G)$, where F and G are functions from $R \times R$ to R. Further, these results are used to obtain an existence theorem for product integrals.

All integrals and definitions are of the subdivision-refinement type, and functions are from the subset $\{(x, y): x < y\}$ of $R \times R$ to R, where R represents the set of real numbers. If $D = \{x_q\}_0^n$ is a subdivision of $[a, b]$ and G is a function, then $D(I) = \{[x_{q-1}, x_q]\}_1^n$ and $G_q = G(x_{q-1}, x_q)$. The statements that G is bounded, $G \in OP^\circ$, $G \in OQ^\circ$ and $G \in OB^\circ$ on $[a, b]$ mean there exist a subdivision D of $[a, b]$ and a positive number B such that if $J = \{x_q\}_0^n$ is a refinement of D, then

(1) $|G(u)| < B$ for $u \in J(I)$,

(2) $|\Pi_r^s(1+G_q)| < B$ for $1 \leqq r \leqq s \leqq n$,

(3) $|\Pi_r^s(1+G_q)| > B$ for $1 \leqq r \leqq s \leqq n$, and

(4) $\Sigma_{J(I)}|G| < B$,

respectively. The notation $\{x_{qr}\}_0^{n(q)}$ represents a subdivision of an interval $[x_{q-1}, x_q]$ defined by a subdivision $\{x_q\}_0^n$. If G is a function, then $G \in S_1$ on $[a, b]$ only if $\lim_{x,y\to p^+} G(x, y)$ and $\lim_{x,y\to p^-} G(x, y)$ exist and are zero for $p \in [a, b]$, and $G \in S_2$ on $[a, b]$ only if $\lim_{x\to p^+} G(p, x)$ and $\lim_{x\to p^-} G(x, p)$ exist for $p \in [a, b]$. Further, $G \in OA^\circ$ on $[a, b]$ only if $\int_a^b G$ exists and $\int_a^b \left|G - \int G\right| = 0$, and $G \in OM^\circ$ on $[a, b]$ only if ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$ and $\int_a^b |1 + G - \Pi(1+G)| = 0$. Also, $G \in OQ^1$ and $G \in OB^*$ on $[a, b]$ if there exists a subdivision $D = \{x_q\}_0^n$ of $[a, b]$ such that

(1) if $1 \leqq q \leqq n$ and $x_{q-1} < x < y < x_q$, then $G \in OQ^\circ$ on $[x, y]$, and

(2) if $1 \leqq q \leqq n$, then either $G \in OB^\circ$ on $[x_{q-1}, x_q]$ or $G - 1 \in OB^\circ$ on $[x_{q-1}, x_q]$,

respectively. The statement that G is almost bounded above by β (or, almost bounded below by β) on $[a, b]$ means there exists a positive integer N such that if D is a subdivision of $[a, b]$ and $u \in H$ only if $u \in D(I)$ and $G(u) > \beta$ (or, $G(u) < \beta$) then H has less than N elements. Consult B. W. Helton [2] and J. S. MacNerney [4] for