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ATOMS ON THE ROYDEN BOUNDARY

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Let R be a hyperbolic Riemann surface and P a nonnegative C^1-density on R. Every $\widetilde{PE}$-minimal function is shown to be $\widetilde{PD}$-minimal. Conversely $\widetilde{PD}$-minimal functions corresponding to atoms in a certain subset Δ_p of the Royden harmonic boundary are $\widetilde{PE}$-minimal. Points in Δ_P are atoms with respect to the PD-representing measure if and only if they are atoms with respect to the HD-representing measure.

Throughout this paper R denotes a hyperbolic Riemann surface. A positive function f in a family of real-valued functions X on R is called *X-minimal* if for every $g \in X$ with $f \geqq g \geqq 0$ there is a constant $c = c(g)$ such that $cf = g$. If Y is any family of functions on R, then the symbol $\tilde{Y}$ is used to denote the functions that are expressible as decreasing limits of sequences of nonnegative functions in Y. The space of harmonic functions with finite Dirichlet integrals over R, $\int_R du \wedge {}^*du < \infty$, is denoted by $HD(R)$ and for a nonnegative C^1-density P on R the space of Dirichlet finite $\left(\text{energy finite, } \int_R du \wedge {}^*du + u^2P < \infty\text{, resp.}\right)$ solutions of the equation $\Delta u = Pu$ on R is denoted by $PD(R)$ ($PE(R)$, resp.)

The study of the spaces $PE(R)$ and $PD(R)$ was initiated by M. Ozawa [9] and H. Royden [10] and recently revitalized by the idea of looking at them in terms of their boundary values on the Royden *harmonic* boundary (cf. [2] and [7]). Following M. Nakai [4] the more general classes $\widetilde{PE}(R)$ and $\widetilde{PD}(R)$ can also be characterized in terms of their boundary values.

One of the main concerns in the study of solutions of $\Delta u = Pu$ on Riemann surfaces is the "comparison theorems" between various spaces of solutions and harmonic function. The purpose of this paper is to give the precise relations between minimal functions in the classes $\widetilde{HD}(R)$, $\widetilde{PE}(R)$, and $\widetilde{PD}(R)$. The relation between the first two notions was given in [1].

Our main results appear in Nos. 7 and 11. Their proofs depend heavily on the results of several papers listed among the references. For the sake of convenience we quote them in Nos. 2 and 3. The results in Nos. 5, 6, and 8 are generalizations of results of M. Nakai for the case $P \equiv 0$. Comparison with the exact references given there and with the exposition in the monograph of Sario-Nakai [11] should clarify what is involved.