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NONSOLVABLE FINITE GROUPS ALL OF WHOSE LOCAL SUBGROUPS ARE SOLVABLE, V

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The earlier papers in this series have reduced the problem of characterizing the minimal simple groups to several stubborn special cases. This paper handles some of these special cases. Almost all the difficulties of this paper involve groups of order $2^a.3^b$ for which $\min(a,b)$ is rather small.

This paper is a continuation of its predecessors.[1] All of the results of this paper are proved on the hypothesis that $2 \in \pi_4$, and most of the results are proved on the additional hypothesis that $e = 2$.

For $i = 0, 1, 2$, let σ_i be the set of all odd primes p in $\pi(\mathfrak{G})$ such that $e(p) = i$. By Theorem 13.8, $\pi(\mathfrak{G}) = \{2\} \cup \sigma_0 \cup \sigma_1 \cup \sigma_2$.

We first record some useful results about π_2.

LEMMA 14.1. *Suppose $p \in \pi_2$ and $\mathfrak{P}$ is a S_p-subgroup $\mathfrak{G}$. If $\mathfrak{P}' \neq 1$, then $\boldsymbol{Z}(\mathfrak{P})$ is cyclic, and in addition, for each non central subgroup $\mathfrak{A}$ of $\mathfrak{P}$ of order p, $\Omega_1(\boldsymbol{C}_{\mathfrak{P}}(\mathfrak{A})) = \mathfrak{A} \times \mathfrak{X}$, where $\mathfrak{X} = \Omega_1(\boldsymbol{Z}(\mathfrak{P}))$ and $\boldsymbol{A}_{\mathfrak{P}}(\mathfrak{A}\mathfrak{X}) = \boldsymbol{A}(\mathscr{C})$, where $\mathscr{C}$ is the chain $\mathfrak{A}\mathfrak{X} \supset \mathfrak{X} \supset 1$.*

Proof. Since $2 \in \pi_4$, p is odd. Suppose $\boldsymbol{Z}(\mathfrak{P})$ is non cyclic. Since $\mathfrak{P}$ has no elementary subgroup of order p^3, $\mathfrak{P}$ has exactly $1 + p$ subgroups of order p, each of which is central. By Theorem 3.2 of [5], $\mathfrak{P}$ is metacyclic. By 0.3.8, $\mathfrak{P}' = 1$.

Suppose $\mathfrak{P}' \neq 1$. By the preceding paragraph, $\mathfrak{X} = \Omega_1(Z(\mathfrak{P}))$ is of order p. Let $\mathfrak{A}$ be a non central subgroup of $\mathfrak{P}$ of order p. Let $\mathfrak{C} = \boldsymbol{C}_{\mathfrak{P}}(\mathfrak{A})$. Since $\mathfrak{A} \neq \mathfrak{X}$, $\mathfrak{C}$ is a proper subgroup of $\mathfrak{P}$. Also, $\mathfrak{A}\mathfrak{X} \subseteq \Omega_1(\mathfrak{C})$. Since $\mathfrak{C}$ has no elementary subgroup of order p^3, we get $\mathfrak{A}\mathfrak{X} = \Omega_1(\mathfrak{C})$ char $\mathfrak{C}$. Since $\mathfrak{C} \subset \mathfrak{P}$, it follows that $\boldsymbol{A}_{\mathfrak{P}}(\mathfrak{A}\mathfrak{X}) \neq 1$. Since $\boldsymbol{A}_{\mathfrak{P}}(\mathfrak{A}\mathfrak{X})$ stabilizes $\mathfrak{A}\mathfrak{X} \supset \mathfrak{X} \supset 1$, the lemma follows.

LEMMA 14.2. *If $p \in \pi_2$ and $p \geqslant 5$, then every p-solvable subgroup of $\mathfrak{G}$ has p-length at most 1. If $\mathfrak{P}$ is a S_p-subgroup of $\mathfrak{G}$, then elements of $\mathfrak{P}$ are $\mathfrak{G}$-conjugate only if they are $\boldsymbol{N}(\mathfrak{P})$-conjugate.*

Proof. We may assume that $\mathfrak{P}' \neq 1$. By 0.4.4, it follows that

[1] Non solvable Finite groups all of whose local subgroups are solvable, I-IV; Bull. Amer. Math. Soc., vol. **74**, no 3, May, (1968), 383-437; Pacific J. Math., vol. **33**, no. 2, (1970), 451-536; Pacific J. Math., vol. **39**, no. 2, (1971), 483-534.