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# AN EXTENSION OF FENCHEL'S DUALITY THEOREM TO SADDLE FUNCTIONS AND DUAL MINIMAX PROBLEMS

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**Fenchel's Duality Theorem (or more precisely, Rockafellar's extension of it) is extended here from the context of convex functions and dual convex extremum problems to that of saddle functions and dual minimax problems. The paper is written in the spirit of mathematical programming. Inequalities between optimal values are established, stable optimal solutions are characterized, strong duality theorems proved, and an existence criterion given. An associated Lagrangian saddle point problem is introduced and an extension of the Kuhn-Tucker Theorem derived. The proofs, which are necessarily different from the purely convex case, rely on recently developed pairs of dual operations on saddle functions, as well as on more widely known facts about conjugate saddle functions.**

1. Introduction. In 1951, Fenchel [1] proved a fundamental and beautiful duality theorem linking the two convex extremum problems

$$\min_x \{f(x) - g(x)\}$$

and

$$\max_{x^*} \{g^*(x^*) - f^*(x^*)\} ,$$

where the function $f$ is proper convex on $R^n$ with conjugate $f^*$, and the function $g$ is proper concave on $R^n$ with conjugate $g^*$. The framework provided by this pair of problems allows one, by suitable choice of the functions $f$ and $g$, to deduce duality results for many different constrained convex extremum problems. Since Fenchel's original result, various refinements and extensions of his theorem have been given by a number of authors. Important among these extensions is the one due to Rockafellar [6] (see also [8, §31]). Besides extending the setting from $R^n$ to arbitrary paired locally convex Hausdorff topological vector spaces, it broadens Fenchel's model problems to include a continuous linear transformation $A$ and its adjoint $A^*$, so that the problems become

$$\min_x \{f(x) - g(Ax)\}$$

and

$$\max_{y^*} \{g^*(y^*) - f^*(A^*y^*)\} .$$