PACIFIC JOURNAL OF MATHEMATICS
Vol. 50, No. 1, 1974

OPEN PROJECTIONS AND BOREL STRUCTURES FOR C^*-ALGEBRAS

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In this paper the relationships existing among the Boolean σ-algebra generated by the open central projections of the enveloping von Neumann algebra $\mathscr{B}$ of a C^*-algebra $\mathscr{A}$, the Borel structure induced by a natural topology on the quasi-spectrum of $\mathscr{A}$, and the type of $\mathscr{A}$ are discussed. The natural topology is the hull-kernel topology. It is shown that this topology is induced by the open central projections and is the quotient topology of the factor states of $\mathscr{A}$ (with the relativized w^*-topology) under the relation of quasi-equivalence. The Borel field is shown to be Borel isomorphic with the Boolean σ-algebra multiplied by the least upper bound of all minimal central projections. Finally, it is shown that $\mathscr{A}$ is GCR if and only if the Boolean σ-algebra (resp. algebra) contains all minimal projections in the center of $\mathscr{B}$, or equivalently, if and only if every point in the quasi-spectrum is a Borel set.

T. Digernes and the present author [10] showed that $\mathscr{A}$ is CCR if and only if the open projections are strongly dense in the center of $\mathscr{B}$. They also showed that the complete Boolean algebra generated by the open central projections is equal to the set of all central projections in $\mathscr{B}$ whenever $\mathscr{A}$ is GCR. Recently, T. Digernes [9] obtained the converse of this result for separable C^*-algebras.

2. **The Boolean algebra of open projections.** Let $\mathscr{B}$ be a von Neumann algebra with center $\mathscr{Z}$ and let $\mathscr{A}$ be a uniformly closed *-subalgebra of $\mathscr{B}$. A projections P in $\mathscr{Z}$ is said to be *open relative to* $\mathscr{A}$ if there is a two-sided ideal $\mathscr{J}$ in $\mathscr{A}$ whose strong closure is $\mathscr{B}P$. In the sequel *all ideals* (unless specifically excluded) will be assumed to be closed two-sided ideals. The definition corresponds to the definition of Akemann [1, Definition II.1] for C^*-algebras with identity. The set $\mathscr{P}(\mathscr{B}, \mathscr{A})$ of all open central projections of $\mathscr{B}$ relative to $\mathscr{A}$ contains 0,1 and the least upper bound (resp. greatest lower bound) of any (resp. any finite) subset [1, Proposition II.5, Theorem II.7].

Now let $\mathscr{B}$ be the enveloping von Neumann algebra of $\mathscr{A}$. The algebra $\mathscr{A}$ will be identified with its embedded image in $\mathscr{B}$. In this case the set $\mathscr{P}(\mathscr{B}, \mathscr{A})$ will be denoted simply by $\mathscr{P}$ and the projection in $\mathscr{P}$ will be called *open* projections. The smallest Boolean algebra (resp. σ-algebra) containing $\mathscr{P}$ will be denoted by $\langle\mathscr{P}\rangle$ (resp. $\langle\!\langle\mathscr{P}\rangle\!\rangle$).