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LIPSCHITZ SPACES ON THE SURFACE OF THE UNIT SPHERE IN EUCLIDEAN n-SPACE

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This paper is concerned with defining Lipschitz spaces on Σ_{n-1}, the surface of the unit sphere in R^n. The importance of this example is that Σ_{n-1} is not a group but a symmetric space. One begins with functions in $L_p(\Sigma_{n-1})$, $1 \leqq p \leqq \infty$. Σ_{n-1} is a symmetric space and is related in a natural way to the rotation group $\mathrm{SO}(n)$. One can then use the group $\mathrm{SO}(n)$ to define first and second differences for functions in $L_p(\Sigma_{n-1})$. Such a function is the boundary value of its Poisson integral. This enables one to work with functions which are harmonic. Differences can then be replaced by derivatives.

For a brief historical survey of Lipschitz spaces, the reader is referred to the introduction in Taibleson [18] and to the papers of Nikolskii [9] and Peetre [10]. For this paper, the approach of two people stands out as being of significant importance.

The first is Zygmund [20; Chapter VII]. Zygmund draws upon the results of Hardy and Littlewood [6]. For brevity we consider only the case $0 < \alpha < 1$. Let $f \in L_p[0, 2\pi]$ and be extended periodically, $1 \leqq p < \infty$, and let

$$\omega_p(\delta) = \sup_{0<h<\delta} \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(x+h) - f(x)|^p\,dx\right\}^{1/p} .$$

Then $f \in \Lambda_\alpha^p$ if and only if $\omega_p(\delta) = O(\delta^\alpha)$. For $p = \infty$, let $\omega_\infty(\delta) = \sup |f(x_2) - f(x_1)|$ where the sup is over all x_1, x_2 such that $|x_1 - x_2| \leqq \delta$. Then $f \in \Lambda_\alpha^\infty$ if and only if $\omega_\infty(\delta) = O(\delta^\alpha)$.

An important result is that $u(r, x)$ is the Poisson integral of a function $f \in \Lambda_\alpha^\infty$ if and only if $(\partial/\partial x)u(r, x) = O(\delta^{\alpha-1})$ where $\delta = 1 - r$, uniformly in x as $r \to 1^-$.

The second person is Taibleson [18]. For brevity we consider only the case $0 < \alpha < 1$. Let $f \in L_p(\boldsymbol{R}^n)$, $1 \leqq p \leqq \infty$, and let $||f(x+h) - f(x)||_{p,dx}$ be the L_p norm of $[f(x+h) - f(x)]$ considered as a function of x. Then $f \in \Lambda(\alpha; p, q)$, $1 \leqq q < \infty$, if and only if

$$\left\{\int_{R^n} [|h|^{-\alpha} ||f(x+h) - f(x)||_{p,dx}]^q dh/|h|^n\right\}^{1/q} < \infty .$$

An important result is that $f(x, y)$, $0 < y < \infty$, is the Poisson integral of a function $f \in \Lambda(\alpha; p, q)$ if and only if