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INTERSECTIONAL PROPERTIES OF CERTAIN FAMILIES OF COMPACT CONVEX SETS

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Let p and q be integers with $p \geqq q \geqq 2$. A family $\mathfrak{F}$ of compact convex subsets of a finite dimensional linear space is said to have the (p, q)-property if $\mathfrak{F}$ contains at least p sets and from each p sets of $\mathfrak{F}$ some q have a common point. In this paper a family $\mathfrak{F}$ is defined to have the (p, q, k)-property in a n-dimensional normed linear space if $\mathfrak{F}$ has the (p, q)-property and an additional property which is measured by k, with $0 \leqq k \leqq 1$. In some sense k measures the "squareness" of the members of $\mathfrak{F}$. The main result is that if $k > 0$, there exists a positive integer $P_n(p, q, k)$ such that each family $\mathfrak{F}$ with the (p, q, k)-property in a n-dimensional normed linear space can be partitioned into $P_n(p, q, k)$ subfamilies each with a nonempty intersection.

Hadwiger and Debrunner have considered the following question: Is there a positive integer $N(p, q, n)$ such that every finite family $\mathfrak{F}$ of sets in E^n with the (p, q)-property can be partitioned into $N(p, q, n)$ subfamilies each of which has a nonempty intersection?

1. Preliminaries. Let $\mathfrak{F}$ be a family of nonempty subsets of a space X and r a positive integer. The family $\mathfrak{F}$ is said to be r-pierceable if there exists a subset F of X consisting of r or fewer points such that $A \cap F \neq \emptyset$ for all $A \in \mathfrak{F}$. If $\mathfrak{F}$ is r-pierceable for some r, then define $|\mathfrak{F}| = \min\{r: \mathfrak{F}$ is r-pierceable$\}$. If $\mathfrak{F}$ is not r-pierceable for any positive integer r, then define $|\mathfrak{F}| = \infty$.

The following lemma is a generalization of a well-known theorem about the intersection of families of closed and compact subsets of X with the finite intersectional property. The proof is routine and is omitted.

LEMMA 1.1. *Let $\mathfrak{F}$ be a family of closed and compact subsets of X and m a positive integer. If $|\mathscr{G}| \leqq m$ for each nonempty finite subfamily $\mathscr{G}$ of $\mathfrak{F}$, then $|\mathfrak{F}| \leqq m$.*

The symbol L^n will denote the n-dimensional normed real linear space consisting of all n-tuples of real numbers whose norm is given by $||(\alpha_1, \cdots, \alpha_n)|| = \max|\alpha_i|$, and the symbol B^n will denote the closed unit ball of L^n. To each ordered pair (x, A) where A is a compact subset of L^n and $x \in A$, associate real numbers $I(x, A)$ and $E(x, A)$ defined by $I(x, A) = \sup\{\lambda: x + \lambda B^n \subset A\}$ and $E(x, A) = \inf\{\lambda \geqq 0: x + \lambda B^n \supset A\}$.