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SOME PROPERTIES OF MODULAR CONJUGATION OPERATOR OF VON NEUMANN ALGEBRAS AND A NON-COMMUTATIVE RADON-NIKODYM THEOREM WITH A CHAIN RULE

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For a cyclic and separating vector Ψ of a von Neumann algebra R, the corresponding modular conjugation operator J_Ψ is characterized by the property that it is an antiunitary involution satisfying $J_\Psi \Psi = \Psi$, $J_\Psi R J_\Psi = R'$ and $(\Psi, Qj_\Psi(Q)\Psi) \geqq 0$ for all $Q \in R$ where $j_\Psi(Q) = J_\Psi Q J_\Psi$.

The strong closure V_Ψ of the vectors $Qj_\Psi(Q)\Psi$ is shown to be a J_Ψ-invariant pointed closed convex cone which algebraically span the Hilbert space H. Any J_Ψ-invariant $\Phi \in H$ has a unique decomposition $\Phi = \Phi_1 - \Phi_2$ such that $\Phi_j \in V_\Psi$ and $s^R(\Phi_1) \perp s^R(\Phi_2)$.

There exists a unique bijective homeomorphism σ_Ψ from the set of all normal linear functionals on R onto V_Ψ such that the expectation functional by the vector $\sigma_\Psi(\rho)$ is ρ. It satisfies

$$\begin{aligned}||\sigma_\Psi(\rho_1) - \sigma_\Psi(\rho_2)||^2 &\leqq ||\rho_1 - \rho_2|| \\ &\leqq \{||\sigma_\Psi(\rho_1) + \rho_\Psi(\rho_2)||\}\,||\sigma_\Psi(\rho_1) - \sigma_\Psi(\rho_2)|| .\end{aligned}$$

Any two σ_Ψ and $\sigma_{\Psi'}$ are related by a unitary u' in R' by $u'\sigma_\Psi(\rho) = \sigma_{\Psi'}(\rho)$ for all ρ.

The relation $l\rho_1 \geqq \rho_2$ holds if and only if there exists $A(\rho_2/\rho_1) \in R$ such that $A(\rho_2/\rho_1)\sigma_\Psi(\rho_1) = \sigma_\Psi(\rho_2)$. The smallest l is given by $||A(\rho_2/\rho_1)||$. It satisfies the chain rule $A(\rho_3/\rho_2)A(\rho_2/\rho_1) = A(\rho_3/\rho_1)$. It coincides with the positive square root of the measure theoretical Radon-Nikodym derivative if R is commutative.

As an application, it is shown that product of any two modular conjugation $j_\Psi j_\Phi$ is an inner automorphism of R.

For a product state $\otimes \rho_j$ of a C^* algebra generated by finite W^* tensor products $\{\otimes_{j \in I} R_j\} \otimes \{\otimes_{j \notin I} 1_j\}$ of von Neumman algebras R_j, it is shown that $\otimes \rho_j$ and $\otimes \rho'_j$ are equivalent if and only if $\Sigma ||\sigma_\Psi(\rho_j) - \sigma_\Psi(\rho'_j)||^2 < \infty$ where $||\sigma_\Psi(\rho) - \sigma_\Psi(\rho')||$ is independent of Ψ.

It is shown that there exists a unitary representation $U_\Psi(g)$ of the group of all $*$-automorphisms of R such that $U_\Psi(g)xU_\Psi(g)^* = g(x)$ for all $x \in R$ and $U_\Psi(g)\sigma_\Psi(g^*\rho) = \sigma_\Psi)\rho)$ for all normal positive linear functionals ρ.

1. Introduction. In the Tomita-Takesaki theory of modular automorphisms [9], two operators Δ_Ψ and J_Ψ are associated with each