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ASSOCIATORS IN SIMPLE ALGEBRAS

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In this paper it is shown that, with suitable hypotheses on the base field, any element of generic trace zero in an octonion algebra is a commutator and an associator, and any element of generic trace zero in a simple Jordan algebra is an associator.

In 1937 K. Shoda [7] proved that every $n \times n$ matrix A of trace zero over a field of characteristic bigger than n is a commutator $[B, C] = BC - CB$ for suitable $n \times n$ matrices B, C. His method was to show that A is similar to a matrix all of whose diagonal entries are zero, and then to give a specific formula for such a matrix. In 1957 A. A. Albert and B. Muckenhoupt [1] proved this theorem for arbitrary fields by proving a more complicated similarity theorem and giving a more complicated formula. In 1963 G. Brown [3] proved an analogous theorem for Lie algebras: that every element of a (split) classical Lie algebra $\mathfrak{L}$ is a commutator. His result is valid over all fields, with the exception of certain small finite fields. His method was to show that if $\mathfrak{L} = \mathfrak{H} \bigoplus \sum_{\alpha \neq 0} \mathfrak{L}_\alpha$ is the Cartan decomposition of $\mathfrak{L}$ with respect to a Cartan subalgebra $\mathfrak{H}$, then every element of $\mathfrak{L}$ is conjugate under the automorphism group of $\mathfrak{L}$ to an element of $\sum_{\alpha \neq 0} \mathfrak{L}_\alpha$.

In this paper we present similar results for alternative and Jordan algebras. If $\mathfrak{A}$ is a (nonassociative) algebra with multiplication $x, y \mapsto xy$, we define the commutator of $x, y \in \mathfrak{A}$ to be $[x, y] = xy - yx$ and the associator of $x, y, z \in \mathfrak{A}$ to be $[x, y, z] = (xy)z - x(yz)$. We prove that in an octonion algebra (with the possible exception of division algebras of characteristic 2) any element of trace zero is both a commutator and an associator. We show that in a simple Jordan algebra over an algebraically closed field of characteristic bigger than the degree of the algebra, every element of trace zero is an associator (the question of commutators does not arise). Our methods are analogous to the above: in each case we prove that an element of trace zero is conjugate under the automorphism group of the algebra to one whose "diagonal entries" (in an appropriate sense) are zero. Then we give a specific computation for such elements.

The product of two elements x, y of an associative or octonion algebra will be denoted xy; the product of elements x, y of a Jordan algebra will be denoted $x.y$. The reader is referred to [2] and [6] for relevant properties of octonion algebras, and to [5], especially Chapter 5, § 6, for properties of simple Jordan algebras.