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A PROBLEM IN COMPACT LIE GROUPS AND FRAMED COBORDISM

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Given a compact, connected, k-dimensional, oriented Lie group G or a faithful orthogonal representation T of such a G there arises an element of the kth framed cobordism group Ω_k^{fr}. The study of these elements is begun, and some algebraic properties of the situation are discussed. The remaining problem is to relate such properties of the elements in Ω_k^{fr} as order or Adams d and e invariants to Lie theory.

If G is a k-dimensional Lie group its tangent bundle may be trivialized by choosing a linear isomorphism of its Lie algebra $\mathscr{L}(G)$ with Euclidean space R^k, and using right multiplication to give an isomorphism of the tangent space at any point with the tangent space at the identity which is, of course, the Lie algebra. If G is compact and oriented every trivialization of the tangent bundle gives rise to a trivialization of the stable normal bundle (see the discussion of tangential and normal structures on p. 23 of [2]) and hence to an element of the kth framed cobordism group Ω_k^{fr}. If two choices of linear isomorphisms of $\mathscr{L}(G)$ with R^k differ by an element of $GL_k(R)$ of positive determinant it is easily seen that the corresponding tangential trivializations are homotopic through trivializations and hence determine the same element of Ω_k^{fr}. Thus, a compact, oriented k-dimensional Lie group G gives rise to a well-defined element $[G] \in \Omega_k^{fr}$.

Now assume in addition that G is connected and let $T: G \rightarrow \mathrm{SO}(n)$ be a faithful representation of G. T embeds G in Euclidean n^2-space. If G is k-dimensional then $k \leqq n(n-1)/2 < n^2/2$, since $\dim G \leqq \dim \mathrm{SO}(n)$, so that codim $G > k$ and the normal bundle of G in this embedding is already stable. We shall always assume a fixed orientation in any Euclidean space we discuss; in particular view Euclidean n^2-space as $M(n)$, the space of $n \times n$ real matrices and choose an orthonormal basis e_{ij}, $1 \leqq i, j \leqq n$, where e_{ij} is the matrix with one in the ijth position and zeroes elsewhere. Orient $M(n)$ by putting the e_{ij} in lexicographic order, so that the ordered basis is $e_{11}, e_{12}, \cdots, e_{1n}, e_{21}, \cdots, e_{2n}, \cdots, e_{n1}, \cdots, e_{nn}$. Make the convention that $M(n)$ is always oriented this way, and also assume that the matrix of any linear transformation from $M(n)$ to itself is always written with respect to this ordered basis.

Returning to the faithful representation T, choose an orthonormal basis $\tau_1, \cdots, \tau_k$ for τ_I the tangent k-plane to $T(G)$ at the identity I.