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ON THE DISTRIBUTION OF NUMBERS OF THE FORM $\sigma(n)/n$ AND ON SOME RELATED QUESTIONS

Dedicated to my friend I. Schoenberg on the occasion of his 70th birthday

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A number theoretic function $f(n)$ is called multiplicative if $f(ab) = f(a)f(b)$ for $(a, b) = 1$, it is called additive if $f(a\,b) = f(a) + f(b)$ for $(a, b) = 1$. A function $f(n)$ is said to have a distribution function if for every c the density $g(c)$ of integers satisfying $f(n) < c$ exists and $g(-\infty) = 0$, $g(\infty) = 1$.

In this note we give some best possible estimates for $g(c + 1/t) - g(t)$, for the case of $f(n) = \sigma(n)/n$.

More than 40 years ago I. Schoenberg proved that $\phi(n)/n$ ($\phi(n)$ is Euler's ϕ function) has a continuous distribution function [12]. This result was the starting point of a systematic theory of additive and multiplicative functions. Very soon Behrend, Chowla, and Davenport [2] proved that $\sigma(n)/n$ ($\sigma(n) = \sum_{d \mid n} d$) also has a continuous distribution function. Thus it followed that the density of abundant numbers $g(2)$ exists. (An integer n if abundant if $\sigma(n)/n \geqq 2$, otherwise it is deficient.) The value $g(2)$ of this density is known only with very poor accuracy, it seems to be fairly close to 1/4 but is not equal to it [1].

I do not discuss here general theory of the distribution of values of additive and multiplicative functions, just remark that necessary and sufficient conditions are known for the existence and continuity of the distribution function of additive and multiplicative functions [4], but relatively little is known about absolute continuity. In 1939, Aurel Wintner called my attention to the problem of absolute continuity of the distribution function of additive and multiplicative functions. I proved (among others) that the distribution function of $\sigma(n)/n$ and $\phi(n)/n$ is purely singular, but that there are additive (and multiplicative) functions whose distribution function is an entire function [5]. No necessary and sufficient condition for the absolute continuity of the distribution function seems to be known and e.g., it is not known if the distribution function of the additive function $f(p) = 1/\log p$ is absolutely continuous.

Denote by $g(c)$ the distribution function of $\sigma(n)/n$. Since $g(c)$ is a purely singular monotonic function its derivative is almost everywhere 0. As far as I know it is not known if the derivative can take any other value. It is easy to see that the derivative from the right of $g(c)$ for $c = \sigma(n)/n$ is infinity, but it is doubtful if the derivative from the left exists. I do not know if the derivative from the right (or left) can take any value other than 0 or infinity. It is easy to see