PACIFIC JOURNAL OF MATHEMATICS
Vol. 52, No. 1, 1974

ON THE EXCEPTIONAL SETS FOR SPACES OF POTENTIALS

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New results on the Bessel and Besov-Lipschitz potentials on R^n are obtained via recent results in nonlinear potential theory. In particular their respective exceptional classes are shown to be identical when $p > 2 - \alpha/n$. By the same techniques, results on thin sets and traces of potentials are obtained.

1. Introduction. In the theory of "perfect functional completion" of a given normed linear space of smooth functions defined on R^n, the idea is to look for a Banach space with respect to the given norm in, say, the class of Lebesgue measurable functions by taking limits in the norm of smooth functions. Associated in a natural way with any such completion is a σ-algebra of exceptional sets of R^n. These exceptional sets give the limits up to which one can pick a canonical equivalence class representative that is defined on the largest possible set. In this note, the exceptional sets for two important perfect functional completions are reexamined in light of recent development in nonlinear potential theory — see e.g., [3], [4], and [7]. The two classes of interest are: $\Lambda_{\alpha,p} = \Lambda_{\alpha\,p}(R^n)$, the Besov-Lipschitz potentials on R^n, and $L_{\alpha,p} = L_{\alpha\,p}(R^n)$, the Bessel potentials on R^n. Their respective exceptional classes are denoted by $\mathfrak{A}^{\alpha,p}$ and $\mathfrak{B}^{\alpha,p}$ in [5], where they are studied extensively — see especially Chapter III page 289 in [5] where a criterion for belonging to $\mathfrak{A}^{\alpha,p}$ or $\mathfrak{B}^{\alpha,p}$ is given. This is utilized in Proposition 1 below.

$L_{\alpha,p}(R^n) = g_\alpha(L_p(R^n))$, i.e., the convolution image of the p-summable functions on R^n, $1 \leqq p \leqq \infty$, under the Bessel kernel $g_\alpha = g_\alpha^{(n)}(x)$, the $L_1(R^n)$ function whose Fourier transform is $(1 + |\xi|^2)^{-\alpha/2}$, $\xi \in R^n$, $\alpha > 0$. The norm on $L_{\alpha,p}$ is $\|u\|_{\alpha\,p} = \|f\|_p$, where $u = g_\alpha * f$ ($\|\cdot\|_p$ the usual norm on L_p). For $\Lambda_{\alpha,p}$, we say $u \in \Lambda_{\alpha\,p}$, $1 \leqq p \leqq \infty$, $0 < \alpha < 1$, if $u \in L_p$ and

$$|u|_{\alpha,p} \equiv \|u\|_p + \left\{\int_{R^n}\int_{R^n}\left(\frac{|\Delta_y u(x)|}{|y|^\alpha}\right)^p \frac{dxdy}{|y|^n}\right\}^{1/p} \tag{1}$$

is finite, $\Delta_y u(x) = u(x-y) - u(x)$. For $1 \leqq \alpha < 2$, $\Delta_y u(x)$ is replaced by $\Delta_y^2 u(x) = u(x-y) + u(x+y) - 2u(x)$ in (1). And finally for $\alpha \geqq 2$, $u \in \Lambda_{\alpha,p}$ iff $u \in L_p$ and $\partial u/\partial x_k \in \Lambda_{\alpha-1,p}$, $k = 1, \cdots, n$. Other equivalent definitions of $\Lambda_{\alpha,p}$ can be found in [9].