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COMPARISON OF THE STATES OF CLOSED LINEAR TRANSFORMATIONS

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Let X and Y be Banach spaces and T, respectively S, be a bounded linear transformation mapping X into Y, respectively Y into X. It is well-known that a nonzero complex number λ belongs to the spectrum of ST precisely when λ belongs to the spectrum of TS. The main result of § 2 shows that for $\lambda \neq 0$ the states of the operators $ST - \lambda I_X$, $TS - \lambda I_Y$ agree.

Sufficient conditions are obtained for this same result to hold when T and S are unbounded closed linear transformations from X into Y and Y into X respectively. Section 4 compares spectral decompositions of ST and TS when these sufficient conditions are satisfied.

Throughout this paper $D(A)$ and $R(A)$ will denote the domain and range of A. The resolvent of A will be denoted $\rho(A)$, the spectrum $\sigma(A)$, the point spectrum $p(A)$ and the approximate point spectrum $a(A)$. $[X, Y]$ will denote the set of all bounded linear transformations, defined on the Banach space X into the Banach space Y. Any other notation used will agree with that of [3]. When no confusion will arise the identity operator will be denoted by I regardless of the space. The following preliminary result can be easily varified.

PROPOSITION 1.1. *If $T: D(T) \subset X \to Y$, $S: D(S) \subset Y \to X$ and $\lambda \neq 0$, then $\lambda \in p(TS)$ if and only if $\lambda \in p(ST)$.*

2. Continuous transformations.

PROPOSITION 2.1. *If $\lambda \neq 0$ then $\overline{R(ST - \lambda I)} = X$ precisely when $\overline{R(TS - \lambda I)} = Y$.*

Proof. $\overline{R(ST - \lambda I)} \neq X$ implies that there exists an $x' \in X'$, $x' \neq 0$ such that $x'((ST - \lambda I)(x)) = 0$ for all $x \in X$. Consequently for all $x \in X$, $0 = (ST - \lambda I)'(x'(x)) = (T'S' - \lambda I)(x'(x))$ and $\lambda \in p(T'S')$. By Proposition 1.1, $\lambda \in p(S'T')$ so $y' \in Y'$, $y' \neq 0$ exists with the property that for each $y \in Y$, $0 = (S'T' - \lambda I)(y'(y)) = y'((TS - \lambda I)(y))$. Thus $\overline{R(TS - \lambda I)} \neq Y$.

The following is a construction of a "generalized" Banach space in the manner of that of Berberian [2].

Denote by glim a fixed "generalized Banach limit" defined for all