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ABIAN'S ORDER RELATION AND ORTHOGONAL COMPLETIONS FOR REDUCED RINGS

W. D. BURGESS AND R. RAPHAEL

Chacron has shown that, in a ring R, the relation "$a \leqq b$ iff $ab = a^2$", first studied by Abian, is an order relation iff R is reduced (has no nilpotent elements). Let R be a reduced ring with 1, a set X in R is *orthogonal* if $ab = 0$ for all $a \neq b$ in X and R is *orthogonally complete* if every orthogonal set in R has a supremum with respect to "$\leqq$". A strongly regular ring is shown to be right (and left) self-injective iff it is orthogonally complete. If $R \subset S$ are reduced rings, S is an *orthogonal extension* of R if every element of S is the supremum of an orthogonal set in R; an orthogonal extension which is complete is an *orthogonal completion*. Completions are unique if they exist. An example shows that not all reduced rings have completions but if R is strongly regular, its complete ring of quotients, $Q(R)$, is its completion. Further, if R is reduced, Baer and such that $Q(R)$ is strongly regular then R has a completion which is a partial ring of quotients.

1. Orthogonal completeness and injectivity. The usual order relation in a Boolean ring extends to reduced rings R when expressed as: $a \leqq b$ iff $ab = a^2$ ([1] and [5]). *In what follows all rings referred to will be reduced* (i.e., 0 is the only nilpotent element) *and with* 1. The basic facts about reduced rings required below can be found in [13] and some of these are quoted here for convenience. If $X \subset R$ then the left and right annihilators of X coincide and will be denoted $\mathrm{Ann}_R X$ or $\mathrm{Ann}\, X$. Also the left and right singular ideals are always trivial and, so, the left and right complete rings of quotients, $Q_l(R)$ and $Q_r(R)$, are always regular. Further, $Q_l(R) = Q_r(R)(= Q(R))$ iff $aR \cap bR = 0$ implies $ab = 0$ for all $a, b \in R$. In this case $Q(R)$ is strongly regular (i.e., $Q(R)$ is also reduced). We note also that all idempotents of a ring R are central and that if R is strongly regular it is duo (i.e., all one-sided ideals are two-sided).

The order relation on a ring R makes R into a partially-ordered multiplicative semigroup since $a \leqq b$ and $c \leqq d$ imply $ac \leqq bd$ ([5]). Also, if $a \leqq b$ in R then $ab = ba$ for $a \leqq b$ implies that $(ab - ba)^2 = 0$. Hence all order properties are right-left symmetric.

In the sequel, if X is a subset of a ring R, $\sup_R X$ or $\sup X$ will always refer to the supremum with respect to "$\leqq$". It is shown in [2] that there is an infinite distributive law in reduced rings. That is, if $X \subset R$ and $\sup X = a$ exists then for any $b \in R$, $\sup bX =$