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RANDOM POINTS IN A SIMPLEX

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The expected values of certain functions of N points chosen at random in an n dimensional simplex or parallelotope are considered, and a decomposition of such integrals is obtained by use of a generalised form of Crofton's Theorem.

Explicit expressions are found for the moments of the area of the triangle formed by three points chosen at random in a triangle or parallelogram.

I. Introduction. In Euclidean n-space a convex body K_n of volume V is given, and $n+1$ points

$$\mathbf{x}_1, \mathbf{x}_2, \dots\dots\dots, \mathbf{x}_{n+1}$$

are chosen independently, at random in K_n.

Let $C(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n+1})$ denote the volume of the convex hull of the $n+1$ points, which with probability one is an n-simplex, and let

$$D(K_n) = C(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n+1})/V.$$

Let $V^h_{K_n}$ be the hth moment of $D(K_n)$

$$V^h_{K_n} = E([D(K_n)]^h).$$

Since a ratio of volumes, and thus a uniform distribution over a body, is preserved under affine transformation, it follows that $V^h_{K_n}$ is an affine invariant of Kn.

The problem of finding $V^h_{K_n}$ is almost trivial for the case $n = 1$, for K_1 is a line segment.

$$V^h_{K_1} = \frac{2}{(h+1)(h+2)}.$$

For the case of $n = 2$, the problem of finding the first moment $V^1_{K_2}$, for various plane convex figures K_2, was investigated by Sylvester and others during the 1860's and has come to be known as Sylvester's Problem (see[3] for references).

For the equivalence class of triangles Δ_2 it is known that