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COMPACT HANKEL OPERATORS AND THE F. AND M. RIESZ THEOREM

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The F. and M. Riesz theorem asserts that every complex Borel measure on the unit circle whose Fourier coefficients with negative index vanish is necessarily absolutely continuous with respect to Lebesgue measure.

The purpose of this note is to give a new proof of Hartman's theorem on compact Hankel operators which clarifies the general context of the theorem. The proof depends only on a few simple operator-theoretic results, Nehari's characterization of bounded Hankel operators, and the aforementioned theorem of F. and M. Riesz.

A Hankel operator on the Hardy space $\mathscr{H}^2$ of functions on the unit circle is an operator of the form

$$H_\phi : f(e^{it}) \to P_+\phi(e^{-it})f(e^{-it})$$

where ϕ is a fixed function in $\mathscr{L}^\infty$ and P_+ is the orthogonal projection of $\mathscr{L}^2$ onto $\mathscr{H}^2$. P. Hartman proved in 1958 that if H_ϕ above is compact, then ϕ can be chosen to be continuous [2].

Let S be the usual unilateral shift on $\mathscr{H}^2$, $S : f(e^{it}) \to e^{it}f(e^{it})$. Nehari had proved earlier that every bounded solution to the operator equation $S^*H = HS$ is of the form $H = H_\phi$, where $\phi \in \mathscr{L}^\infty$, and had further shown that ϕ could be chosen so that $\|H\| = \|\phi\|_\infty$ [3].

R. N. Hevener [3] and D. Sarason [5] have previously used the Riesz theorem in obtaining Hartman's characterization of compact Hankel operators. Their procedure involves the factorization of analytic functions. T. L. Kriete motivated this paper by suggesting that Lemma 1 below might be used to make more precise the relationship between compact Hankel operators and the F. and M. Riesz theorem.

Let $\mathscr{C}$ denote the space of complex-valued continuous functions on the unit circle, and $\mathscr{A}_0$ the subspace of $\mathscr{C}$ consisting of those functions which extend continuously to an analytic function on the unit disk vanishing at the origin. With the F. and M. Riesz theorem and the Riesz representation theorem for bounded linear functionals on $\mathscr{C}$, one easily shows that the Hardy space $\mathscr{H}^1$ is the dual space of $\mathscr{C}/\mathscr{A}_0$. Let $\mathscr{H}_0^\infty$ denote the class of $\mathscr{H}^\infty$ functions which vanish at zero. Since $\mathscr{L}^\infty$ is