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ON A THEOREM OF BRAUER-CARTAN-HUA TYPE

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We shall be concerned here with the nature of subrings of a ring with involution which are invariant with respect to certain combinations of elements. To be more precise, let R be a ring with involution $*$ and suppose that A is a subring of R such that $xAx^* \subset A$ for all $x \in R$. Can we say something definitive about the structure of A? We shall see that if R is semi-prime then we do get a dichotomy of the Brauer-Cartan-Hua type, namely, A must contain a nonzero ideal of R or A must be central.

Considerations of such kind of subrings of R arose in the Ph. D. thesis of P. Lee [2].

In what follows, R will be a semi-prime ring with involution $*$ and A will be a subring of R such that $xAx^* \subset A$ for all $x \in R$.

We begin with

LEMMA 1. *If A does not contain a nonzero ideal of R, then $ab^* = ba$ and $b^*a = ab$ for all $a, b \in A$.*

Proof. Let $a \in A$. Linearize $xax^* \in A$ by replacing x by $x + y$. We get

(1) $xay^* + yax^* \in A$ for all $a \in A$, $x, y \in R$.

In (1) replace x by xb, where $b \in A$. We get

(2) $xbay^* + yab^*x^* \in A$.

However, by (1), since $ba \in A$

(3) $x(ba)y^* + ybax^* \in A$.

Subtracting (3) from (1) gives $y(ab^* - ba)x^* \in A$ for all $x, y \in R$, hence $R(ab^* - ba)R \subset A$.

Since A does not contain a nonzero ideal of R, but $A \supset R(ab^* - ba)R$, we deduce that $R(ab^* - ba)R = 0$. However, since R is semi-prime, we conclude that $ab^* - ba = 0$, and so $ab^* = ba$.

If we use a similar argument, replacing y by yb^*, $b \in A$, in (1) we end up with the other relation, $b^*a = ab$.

From Lemma 1 we can settle the problem for A noncommutative.

LEMMA 2. *If A is noncommutative and $xAx^* \subset A$ for all $x \in R$ then A contains a nonzero ideal of R.*

Proof. Suppose the conclusion of the lemma is false. Then, by