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STRONGLY SUPERFICIAL ELEMENTS

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The concepts of a strongly superficial element and a very strongly superficial element are introduced. A number of their properties are established and three applications are given.

1. Introduction. Superficial elements have proved to be a useful and important concept in a number of problems in commutative algebra, for example, the study of characteristic functions and multiplicities. This paper is concerned with two special kinds of such elements: a very strongly superficial (v.s.s.) element of degree k for an ideal A in a ring R; and, a strongly superficial (s.s.) element for A^k. After listing a number of properties of s.s. and v.s.s. elements, we present in Theorem (2.5) and (2.6) a number of characterizations of such elements. In §3 we give three applications of the theorems. Namely, we first show that a known result about s.s. elements for an ideal generated by an R-sequence in a locally Macaulay ring holds in every Noetherian ring (3.2). Next we show that if A is an ideal in a Noetherian ring R, then the zero ideal in the A-form ring of R has no irrelevant prime divisor if and only if there exists a v.s.s. element of some positive degree for A (3.5). The final application is concerned with certain ideals in Rees rings of R ((3.8) and (3.9)).

2. s.s. and v.s.s. elements. All rings in this paper are assumed to be commutative with a unit element.

DEFINITION. 2.1. Let A be an ideal in a ring R, and let k be a positive integer. A *superficial element of degree k for A* is an element $x \in A^k$ for which there exists a nonnegative integer c such that $(A^{n+k}: xR) \cap A^c = A^n$, for all integers $n \geqq c$. If $c = 0$ (where $A^0 = R$), then x is said to be a *very strongly superficial* (v.s.s.) *element of degree k for a*. If $A^{nk}: xR = A^{nk-k}$, for all integers $n \geqq 1$, then x is said to be a *strongly superficial* (s.s) *element for A^k*.

It is easily seen that, if $A^n \neq A^{n+1}$ (for each integer $n \geqq 0$) and x is a superficial element of degree k for A, then $x \notin A^{k+1}$. (In particular, a v.s.s. element of degree k for A is not in A^{k+1}.) It is also clear that a v.s.s. element of degree k for A is a s.s. element for A^k. Some further properties of such elements are given in the following remark.

REMARK 2.2. Let A be an ideal in a Noetherian ring R, let k be a positive integer, and assume x is a s.s. element for A^k.