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THE UNIVERSAL FLIP MATRIX AND THE GENERALIZED FARO-SHUFFLE

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The permutation matrix P which flips a left direct product of two matrices into a right direct product is investigated on the basis of the generalized out faro-shuffle. Expressions are derived for the characteristic and minimal polynomials as well as the determinant and trace of P.

1. Introduction. One of the most interesting and lesser known matrices in the theory of left direct products, [7] p. 81, is the permutation matrix P, whose (i, j) block entry is the $m \times n$ unit matrix $E_{ji} = [\delta_{si}\delta_{rj}]$, $i = 1, \cdots, n$ $j = 1, \cdots, m$ namely

$$P = P_{m,n} = \begin{bmatrix} E_{11} E_{21} \cdots E_{m1} \\ E_{12} \quad\quad \vdots \\ \vdots \quad\quad\quad \vdots \\ E_{1n} \cdots\cdots E_{mn} \end{bmatrix} = P^T_{n,m}.$$

The reason being, that for *any* pair of matrices $A_{m\times m}$, $B_{n\times n}$

$$P(A \otimes B)P^{-1} = B \otimes A, \tag{1}$$

i.e. P flips the order in any left direct product or equivalently converts a left direct product into a right direct product. If A and B are finite group representations then P represents the isomorphism relating the direct products of the two groups. In this note we give a short matrix proof of this result, which is valid over any commutative ring R with identity 1, and derive some further properties of the matrix P, after identifying the permutation Π associated with P with the generalized out faro-shuffle for a 1-dimensional deck of mn cards [9] [2]. In particular we shall give expressions for the characteristic and minimal polynomials of P in terms of m and n.

Throughout this paper we assume that all our matrices are over a commutative ring R with unity and char $R \neq 2$. Whenever necessary we shall first prove the result for real matrices, then specialize to integer matrices and employ Theorem 11 of [8], p. 49, to obtain the corresponding result for commutative rings with unity. In fact when working with matrices whose entries are integer multiples of $1 \in R$, we