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FIXED POINT ITERATIONS OF NONEXPANSIVE MAPPINGS

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Let C be a boundedly weakly compact convex subset of a Banach space E. Suppose that each weakly compact convex subset of C possesses the fixed point property for nonexpansive mappings, and let $T\colon C \to C$ be nonexpansive. In this note it is shown (by a very simple argument) that if a sequence of iterates of T (generated with the aid of an infinite, lower triangular, regular row-stochastic matrix) is bounded, then T has a fixed point.

Dotson and Mann [3] proved this theorem under the additional assumption that E was uniformly convex. (Their complicated proof relied heavily on the uniform convexity of E.) We use our method also to establish a similar result (essentially due to Browder) for nonlinear nonexpansive semigroups.

Let C be a closed convex subset of a Banach space $(E, |\ \ |)$, and let $T\colon C \to C$ be nonexpansive (that is, $|Tx - Ty| \leqq |x - y|$ for all x and y in C). Let N denote the set of nonnegative integers, and suppose $A = \{a_{nk}\colon n, k \in N\}$ is an infinite matrix satisfying

$$\begin{aligned} a_{nk} &\geqq 0 \quad \text{for all} \quad n, k \in N, \\ a_{nk} &= 0 \quad \text{if} \quad k > n, \\ \sum_{k=0}^{n} a_{nk} &= 1 \quad \text{for all} \quad n \in N, \\ \lim_{n\to\infty} a_{nk} &= 0 \quad \text{for all} \quad k \in N. \end{aligned}$$

If x_0 belongs to C, then a sequence $S = \{x_n\colon n \in N\} \subset C$ can be defined inductively by

$$x_n = a_{n0}\, x_0 + \sum_{k=1}^{n} a_{nk}\, T x_{k-1}, \qquad n \in N.$$

This iteration scheme is due to Mann [8].

It is not difficult to see that if T has a fixed point, then S is bounded. Dotson and Mann [3, Theorem 1] have proved that if E is uniformly convex, and if S is bounded for some x_0 in C, then T has a