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ON THE LATTICE OF NORMAL SUBGROUPS OF A DIRECT PRODUCT

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Suzuki has determined that if G is a direct product $G = \Pi_{i=1}^{k} G_i$ of groups $G_i \neq 1$, then the lattice $L(G)$ of subgroups of G is the direct product of the lattices $L(G_i)$ if and only if the order of any element in G_i is finite and relatively prime to the order of any element in $G_j (i \neq j)$. An exercise in Zassenhaus' *The Theory of Groups* asks the reader to prove an analogous result for the lattice of normal subgroups. In §1, we derive this result for the case of the direct product of two groups. (The generalization to the direct product of any finite number of groups is straightforward.) In §2, we use results obtained in §1 to study in detail the normal subgroup lattice of the direct product of finitely many symmetric groups.

1. The lattice of normal subgroups. If G_1 and G_2 are groups, we denote elements of the direct product $G_1 \times G_2$ by ordered pairs (a, b), $a \in G_1$, $b \in G_2$. If A and B are subgroups of a group G, we define $[A, B] = \langle aba^{-1}b^{-1} \mid a \in A, b \in B \rangle$, and note that if $A \lhd G$, then $[A, B] \lhd A$. We let ρ_1 and ρ_2 denote the first and second projection maps on $G_1 \times G_2$, and finally, we denote by $o(g)$ the order of the element g.

If N is a subgroup of $G_1 \times G_2$, we put $N_1 = \rho_1(N)$ and $N_2 = \rho_2(N)$. Thus N_i is a subgroup of G_i, called the ith projection of N. Furthermore, if $N \lhd G_1 \times G_2$, then $N_i \lhd G_i$.

LEMMA 1. *If $N \lhd G_1 \times G_2$, then $N \supseteq [G_1, N_1] \times [G_2, N_2]$.*

Proof. Let $a \in N_1$. Then there exists $y \in N_2$ such that $(a, y) \in N$. Thus $(a^{-1}, y^{-1}) \in N$, and since $N \lhd G_1 \times G_2$, $(g, 1)(a, y)(g^{-1}, 1) = (gag^{-1}, y) \in N$. It follows that $(gag^{-1}, y)(a^{-1}, y^{-1}) = (gag^{-1}a^{-1}, 1) \in N$, so $N \supseteq [G_1, N_1] \times \{1\}$. Similarly, $N \supseteq \{1\} \times [G_2, N_2]$, completing the proof.

The following lemma, whose proof is immediate, will be used in the discussion that follows.

LEMMA 2. *Let G be a group, $H \lhd G$. Then any subgroup L of G such that $[G, H] \subseteq L \subseteq H$ is normal in G.*

Since $[G_1 \times G_2, A \times B] = [G_1, A] \times [G_2, B]$ whenever $A \subseteq G_1$, $B \subseteq G_2$, if $N \lhd G_1 \times G_2$ with projections N_1 and N_2, then any subgroup of $G_1 \times G_2$ lying between $[G_1, N_1] \times [G_2, N_2]$ and $N_1 \times N_2$ is normal in