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ON LINEAR REPRESENTATIONS OF AFFINE GROUPS I

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The category of linear representations of an affine group is isomorphic to the category of comodules over a k-Hopf-algebra where k denotes a commutative ring. The category of C-comodules Comod-C over an arbitrary k-coalgebra C is comonadic over the category k-Mod of k-modules. It is complete, cocomplete and has a cogenerator. The C-comodules whose cardinality $\leqq$ max (cardk, $\aleph_0$) generate the category Comod-C. Comod-C is in general not abelian but can nicely be embedded into an AB-4 category. Comod-C is a tensored and cotensored k-Mod-category (enriched over k-Mod) with a canonical (E, M)-factorization which is the factorization in k-mod if and only if C is flat. Comod-C has free C-comodules if and only if C is finitely generated and projective. Furthermore I give numerous examples and counter-examples as well as the explicit description of all constructions, in particular of the limits in Comod-C which was not known even for coalgebras over fields.

Let k be a commutative ring with a unit. k-Alg shall denote a small category of models of k-algebras (cf. [5] p. XXIV). Recall that an affine k-monoid (resp. k-group) is a monoid (resp. group) in the functor category [k-Alg, Sets] whose underlying functor is representable. Let M be a k-module. Then M induces an affine k-monoid $\mathscr{L}(M)$: k-Alg $\rightarrow$ Sets by $\mathscr{L}(M)(A) = \mathrm{End}_A(M \otimes_k A)$, $A \in$ k-Alg (cf. [5] p. 149). Let $\mathscr{G}$ be an affine k-monoid and M a k-module. Then a monoid morphism $\varphi: \mathscr{G} \rightarrow \mathscr{L}(M)$ is called a linear representation of $\mathscr{G}$ in M and the pair (M, φ) a k-$\mathscr{G}$-module. The definition of morphisms between k-$\mathscr{G}$-modules is evident. Thus one obtains the category k-$\mathscr{G}$-Mod of linear representations of $\mathscr{G}$, resp. of k-$\mathscr{G}$-modules. Since $\mathscr{G}$ is representable we obtain the canonical isomorphisms [k-Alg, Sets] $(\mathscr{G}, \mathscr{L}(M)) \cong \mathscr{L}(M)(C) \cong k$-Mod $(M, M \otimes_k C)$, where C is the representing object of $\mathscr{G}$. The monoid structure of $\mathscr{G}$ induces a k-coalgebra structure on C, i.e., the representing object has two k-linear mappings $\Delta: C \rightarrow C \otimes C$ and $\varepsilon: C \rightarrow k$, called comultiplication and counit, such that $\langle C, \Delta, \varepsilon \rangle$ is coassociative and counitary (cf. [19]). By the above canonical isomorphisms every monoid morphism $\varphi: \mathscr{G} \rightarrow \mathscr{L}(M)$ induces a k-linear map $\chi_M: M \rightarrow M \otimes C$ such that $M \otimes \Delta \cdot \chi_M = \chi_M \otimes C \cdot \chi_M$ and $M \otimes \varepsilon \cdot \chi_M = \mathrm{id}_M$, and conversely. A pair $\langle M, \chi_M \rangle$ fulfilling the above properties is called a C-comodule. Let $\langle M, \chi_M \rangle$ and $\langle N, \chi_N \rangle$ be C-comodules. A k-linear mapping $f: M \rightarrow N$ is a C-comodule homo-