PACIFIC JOURNAL OF MATHEMATICS
Vol. 61, No. 2, 1975

COMMUTATIVE CANCELLATIVE SEMIGROUPS WITHOUT IDEMPOTENTS

H. B. HAMILTON, T. E. NORDAHL AND T. TAMURA

A commutative cancellative idempotent-free semigroup (CCIF-) S can be described in terms of a commutative cancellative semigroup C with identity, an ideal of C, and a function of $C \times C$ into integers. If C is an abelian group, S has an archimedean component as an ideal; S is called an $\bar{\mathfrak{N}}$-semigroup. A CCIF-semigroup of finite rank has nontrivial homomorphism into nonnegative real numbers.

1. **Introduction.** In this paper, a commutative cancellative semigroup without idempotent is called a CCIF-semigroup (in which, by "IF" we mean "idempotent-free") and a commutative cancellative semigroup with identity is called a CCI-semigroup. In particular, an $\mathfrak{N}$-semigroup is an archimedean CCIF-semigroup. The structure of $\mathfrak{N}$-semigroups has been much studied [1, 2, 3, 6, 7, 8] and also it is well known that every CCIF-semigroup is a semilattic of $\mathfrak{N}$-semigroups. In this paper CCIF-semigroups will be studied by means of the representation by the generalized $\mathscr{I}$- and φ-functions and also through homomorphisms into the nonnegative real numbers.

Throughout this paper, $\boldsymbol{R}$ denotes the set of real numbers; R the set of rational numbers; $\boldsymbol{R}_+$ the set of positive real numbers; $\boldsymbol{R}_+^0$ the set of nonnegative real numbers; Z_+ the set of positive integers and Z_+^0 the set of nonnegative integers. Each of these is a semigroup under the usual addition. If S is a semigroup and if X is a subsemigroup of the group $\boldsymbol{R}$, then the notation $\mathrm{Hom}(S, X)$ denotes the semigroup of homomorphisms of S into X under the usual operation.

At the end of §1 we show that if S is a CCIF-semigroup, $\mathrm{Hom}(S, \boldsymbol{R}) \neq \{0\}$, and the homomorphism group is transitive in some sense. In Section 2 we shall try to generalize the representation of $\mathfrak{N}$-semigroups to CCIF-semigroups. It will be understood as the so-called Schreier's extension to build up complicated CCIF-semigroups from simpler CCIF-semigroups. Most of the results in [7] will be extended to CCIF-semigroups. In §3 we shall treat the important case, i.e., the case where the structure semigroup is a group. Such a CCIF-semigroup will be called an $\bar{\mathfrak{N}}$-semigroup. In §4 we shall show that every CCIF-semigroup of finite rank has a nontrivial homomorphism into $\boldsymbol{R}_+^0$. In particular we will characterize CCIF-semigroups S having the property $\mathrm{Hom}(S, \boldsymbol{R}_+) \neq \varnothing$.

(1.1) *Let S be a* CCIF-*semigroup. Then $x \neq xy$ for all $x, y \in S$.*