PACIFIC JOURNAL OF MATHEMATICS
Vol. 61, No. 2, 1975

PERMUTATION POLYNOMIALS OVER THE RATIONAL NUMBERS

CLIFTON E. CORZATT

Nonlinear polynomials, over the rational numbers, which permute the integers $0, 1, \cdots N$ are investigated. The function $\nu(N)$ represents the minimum degree of all such polynomials. It is shown that

$$\left[\frac{N+1}{4}\right] \leqq \nu(N) \leqq N-1 \quad \textbf{for all} \quad N \geqq 13 \, .$$

It is also shown that $\nu(N) \leqq N-2$ for N odd and $N \geqq 7$, that $\nu(N) \leqq N-3$ for $N = 2 \bmod 6$, and that if $\varepsilon > 0$ then $\nu(N) \geqq ((N-1)/2)(1-\varepsilon)$ for N sufficiently large.

1. Introduction. We wish to study polynomials with rational coefficients which permute the integers $0, 1, \cdots, N$. Specifically, if we fix N, then are we able to find nonlinear polynomials of this type which have degree less than N? If so, how small can the degree of such a polynomial be? If $N > 4$ we will show that there are polynomials whose degree is less than N. For certain infinite classes of integers we can show that there are polynomials whose degree is less than $N-1$ and $N-2$. Moreover, we show that if $\varepsilon > 0$ then for N sufficiently large the degree of such a polynomial is bounded below by $(N-1)(1-\varepsilon)/2$.

This problem was suggested by Professor L. A. Rubel and arose in the following context. Polya showed that if an entire analytic function of exponential type less than log 2 has integer values at each nonnegative integer, then it is a polynomial. A proof of this theorem is given on page 175 of *Entire Functions* by R. P. Boas. Rubel conjectures that if an entire analytic function of exponential type less than π permutes the nonnegative integers then it is the function $f(z) = z$. He gives the function $f(z) = z + \cos(\pi z)$ as an example of an entire analytic function of exponential type π which permutes the nonnegative integers.

The problem which we study here is an analogue in which we assume $f(z)$ is a polynomial and that it permutes only the integers $0, 1, \cdots, N$. We show that the degree of the polynomial is fairly large with respect to N or it is of degree 1. Rubel's conjecture says that an entire analytic function which permutes the nonnegative integers is of relatively large exponential type (compared to log 2) or it is a polynomial of degree 1. As far as we know this work bears no relationship to the extensive collection of papers which