PACIFIC JOURNAL OF MATHEMATICS
Vol. 61, No. 2, 1975

PROPERTIES WHICH NORMAL OPERATORS SHARE WITH NORMAL DERIVATIONS AND RELATED OPERATORS

JOEL ANDERSON AND CIPRIAN FOIAŞ

Let S and T be (bounded) scalar operators on a Banach space $\mathscr{X}$ and let $C(T, S)$ be the map on $\mathscr{B}(\mathscr{X})$, the bounded linear operators on $\mathscr{X}$, defined by

$$C(T, S)(X) = TX - XS$$

for X in $\mathscr{B}(\mathscr{X})$. This paper was motivated by the question: to what extent does $C(T, S)$ behave like a normal operator on Hilbert space? It will be shown that $C(T, S)$ does share many of the special properties enjoyed by normal operators. For example it will be shown that the range of $C(T, S)$ meets its null space at a positive angle and that $C(T, S)$ is Hermitian if T and S are Hermitian. However, if $\mathscr{X}$ is a Hilbert space then $C(T, S)$ is a spectral operator if and only if the spectrum of T and the spectrum of S are both finite.

We now indicate our results in greater detail. Let $\mathscr{H}$ be a Hilbert space and let N be a normal operator in $\mathscr{B}(\mathscr{H})$. Then N enjoys the following properties.

(A) $\mathscr{R}(N)$ is orthogonal to $\mathscr{N}(N)$, where $\mathscr{R}(N)(\mathscr{N}(N))$ donotes the range (null space) of N.

(B) $\mathscr{R}(N)^- \oplus \mathscr{N}(N) = \mathscr{H}$, where the bar denotes norm closure.

(C) There is a resolution of the identity $E(\cdot)$ supported by $\sigma(N)$ such that

$$N = \int_{\sigma(N)} \lambda dE ,$$

where $\sigma(N)$ denotes the spectrum of N. That is, N is a scalar operator.

(D) If $x \in E(\{\lambda\})\mathscr{H}$ for some complex number λ, then $Nx = \lambda x$.

(E) N has closed range if and only if 0 is an isolated point in $\sigma(N)$. (We adopt the convention that 0 is isolated in $\sigma(N)$ if $0 \notin \sigma(N)$).

(F) The norm, spectral radius, and numerical radius of N are equal.

(G) The closure of the numerical range of N is the convex hull of the spectrum of N.

In §§ 1, 2, and 3 of this paper we show that appropriate versions of (A), (D), and (E) hold for $C(T, S)$. In Section 4 we restrict ourselves to the Hilbert space case and show that (B) is false in