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FILTERED SPACES ADMITTING SPECTRAL SEQUENCE OPERATIONS

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Introduction. Throughout this paper we will work in the category of compactly generated Hausdorff spaces. All cohomology will be singular cohomology with Z_2 coefficients, although similar results may be derived when 2 is replaced by an odd prime.

We will call a filtration $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X$ of a space X convergent if $X = \cup X_n$ and every compact subset of X is contained in some X_n. In this case the cohomology spectral sequence of X converges. A subspace A of X has the induced filtration $A_k \equiv A \cap X_k$. A map $f: X \to Y$ will be called filtered if for each n, $f(X_n) \subseteq Y_n$. The product of two filtered spaces X and Y is given the "tensor product" filtration $(X \times Y)_n \equiv \bigcup_{i+j=n} X_i \times Y_j$. S^∞ will represent the infinite sphere with regular CW reticulation consisting of two cells in each dimension, with the skeletal filtration, and with the antipodal Z_2 action.

We will define a category of convergent filtered spaces and prove that each object Q has the following properties. First, the diagonal map $Q \xrightarrow{d_Q} Q \times Q$ is homotopic to an essentially unique filtered diagonal approximation. Secondly, consider the map $S^\infty \times Q \xrightarrow{\pi_Q} Q \xrightarrow{d_Q} Q \times Q$, where π_Q represents projection, and Z_2 acts only on the first component of $S^\infty \times Q$ and by transposition on $Q \times Q$. Then $d_Q \circ \pi_Q$ is equivariantly homotopic to an essentially unique filtered equivariant diagonal approximation.

A diagonal approximation can be used in the obvious way to define a product in the spectral sequence of Q. An equivariant diagonal approximation can be used to define operations in the spectral sequence. The latter construction proceeds in direct analogy with the construction of Steenrod squares for regular CW-complexes, replacing the cellular cochain complex of the CW-complex with the E_1 level of the spectral sequence of Q.

The category of convergent filtered spaces to be defined includes (up to filtered homotopy type) the Serre filtration of the total space of a fibration over a regular CW-complex, the Milgram-Dold-Lashof filtration of the classifying space of a topological monoid, and a special case of the filtered spaces used in deriving the fiber-square (Eilenberg-Moore) spectral sequence.

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