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BANACH SPACES WITH A RESTRICTED HAHN-BANACH EXTENSION PROPERTY

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We shall study the class of real Banach spaces B with the following restricted Hahn-Banach extension property: For each Banach space C with a dense set of cardinality $\leqq$ some fixed cardinal $\mathfrak{N}$, and for each subspace A of C and bounded linear map $T_0: A \to B$, there exists an extension $T: C \to B$ such that $\|T\| = \|T_0\|$. Suprisingly, there exist Banach spaces in this class which are not isometrically isomorphic to $C(X)$ for a compact Hausdorff X!

The combined results of Goodner, Hasumi, Kelley and Nachbin show that those Banach spaces with the Hahn-Banach extension property, that is, those Banach spaces which are injective in the category $\mathscr{B}_1$ of Banach spaces and linear maps of norm $\leqq 1$, are precisely the Banach spaces of the form $C(X)$, where X is compact Hansdorff and extremally disconnected [5], [6], [7], [11]. In this paper, we wish to study those Banach spaces which enjoy a restricted Hahn-Banach extension property, where the existence of an extension is only required for spaces which are relatively small.

To be more precise, let $\mathfrak{N}$ be an infinite cardinal. We shall say that a Banach space C is $\mathfrak{N}$-separable if C has a dense subset of cardinality $\mathfrak{N}$. As usual, the word "separable" standing alone means $\mathfrak{N}_0$ separable. We shall call a Banach space B $\mathfrak{N}$-injective if B has the following restricted Hahn-Banach extension property: Let C be an $\mathfrak{N}$-separable Banach space, let A be a subspace of C, let $i: A \hookrightarrow C$ be the inclusion map, and let $T_0: A \to B$ be a bounded linear map. Then there exists a bounded linear map T with $\|T\| = \|T_0\|$, making the following diagram commute:

(1)
$$\begin{tikzcd} A \arrow[rr, hook, "i"] \arrow[dr, "T_0"'] & & C \arrow[dl, "T"] \\ & B & \end{tikzcd}$$

We shall study the $\mathfrak{N}$-injective Banach spaces in this paper. We shall only consider real Banach spaces here. We shall characterize the Banach spaces of type $C(X)$ which are $\mathfrak{N}$-injective. We shall also show that there are a good many other $\mathfrak{N}$-injective Banach spaces! Finally, we shall show that if an $\mathfrak{N}$-injective Banach space also happens to be $\mathfrak{N}$-separable, then it is in fact injective in the full category $\mathscr{B}_1$. This contrasts rather sharply with the situation