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FUNDAMENTAL UNITS AND CYCLES IN THE PERIOD OF REAL QUADRATIC NUMBER FIELDS

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PART I

0. Introduction. In this paper we introduce the concept of "Cycles in the Period" of the simple continued fraction expansion of a real quadratic irrational. This is expressed in the

DEFINITION. Let M, D, d be positive rational integers, M sequare free, $M = D^2 + d, d \leqq 2D$. Let k, a, s be nonnegative rational integers, $0 \leqq a \leqq k - 1$; let $f = f(k, a, s; d, D)$ be a polynomial with rational integral coefficients. For a fixed s, the finite sequence of polynomials

$$(0.1) \qquad F)s) = f(k, a, s; d, D), f(k, a+1, s; d, D), \cdots, \\ f(k, a+k-1, s; d, D)$$

will be called "Cycle in the Period" of the simple continued fraction expansion of $\sqrt{M}$ if, for $s_0 \geqq 1$, this expansion has the form

$$(0.2) \qquad \begin{aligned} \sqrt{M} = & \overline{[b_0, b_1, \cdots, F(0), \cdots, F(s_0-1), f(k, a, s_0; d, D), \cdots,} \\ & \overline{f(k, a+b, s_0; d, D), \cdots, f(k, a, s_0; d, D), F'(s_0-1), \cdots,} \\ & \overline{F'(0), f(k, a-1, 0; d, D), \cdots, b_1, 2b_0]} \end{aligned}$$

$b \geqq 1; b \leqq k - 1$; k is the length of the cycle; $F'(s)$ means that the order of the $f - s$ must be reversed.

In the first part of this paper, the main result is the construction of infinitely many classes of quadratic fields $Q(\sqrt{M})$, each containing infinitely many M of a simple structure. Among the various classes thus constructed, there are a few in whose expansion of $\sqrt{M}$ cycles in the period surprisingly have the length $\leqq 12$. Functions $f(k, a, s; d, D)$, $f(k, a+1, s; d, D), \cdots$ are of course stated explicitly; hence we are able to construct numbers $\sqrt{M}$ such that the primitive period of their expansion has any given length m which is a function of the parameter k.

Expansions of $\sqrt{M}$ which have the structure of cycles in the period were generally not known up to now. In a recent paper Y. Yamamoto [6] has given a few numerical examples of expantions of