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THE EXISTENCE OF NATURAL FIELD STRUCTURES FOR FINITE DIMENSIONAL VECTOR SPACES OVER LOCAL FIELDS

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Let K be a local field (e.g., a p-adic or p-series field) and n a positive integer. Let K' be the unique (up to isomorphism) unramified extension of K. It is shown that the natural (modular) norm of K' is the nth power of the usual (l^∞) vector space norm of K' when K' is viewed as an n-dimensional vector space over K. Further, the two distinct descriptions of the dual of K' (which is isomorphic to K') that arise from the field model and vector space model are isomorphic under a K-linear isomorphism of K' as a vector space over K, and the isomorphism is norm preserving.

1. If $\boldsymbol{R}^n$ is n-dimensional Euclidean space and $n > 1$, then the only case for which $\boldsymbol{R}^n$ has a (commutative) field structure is $n = 2$. In that case $\boldsymbol{R}^2$ can be identified as the additive group of $\boldsymbol{C}$, the complex numbers, and the norms for $\boldsymbol{R}^2$ and $\boldsymbol{C}$ are compatible in the following sense: Let $(x, y) \in \boldsymbol{R}^2$ and consider the correspondence $(x, y) \leftrightarrow z = x + iy$. The norm of $(x, y) \in \boldsymbol{R}^2$ is $|z|_{R^2} = |(x, y)|_{R^2} = (x^2 + y^2)^{1/2}$. Let dz be Haar measure on $\boldsymbol{C}$. We define $N_C(\mathrm{w}) = w\bar{w}$ and $\mathrm{mod}_C(w)$ by the relation $d(wz) = \mathrm{mod}_C(w)dz$. We obtain, as is well known: $|z|_{R^2}^2 = N_C(z) = \mathrm{mod}_C(z)$.

We will show that if K is a local field (e.g., if K is a p-adic field) and n is an integer greater than 1, then K^n, the n-dimensional vector space over K, has a field structure, as a local field, which is compatible with the usual vector space norm of K^n, in the same sense as above.

The reader is referred to [3; Ch. I] for a review of the basic facts about local fields and to [4; Chs. I-II] for many details and proofs.

2. Let K be a local field; which is to say a locally compact, nondiscrete field that is not connected. The K is totally disconnected. Such a field is either a p-adic field, a finite algebraic extension of a p-adic field or the field of formal Laurent series over a finite field. The ring of integers, $\mathfrak{O}$, in K is the unique maximal compact subring of K. The prime ideal, $\mathfrak{O}$, in $\mathfrak{D}$, is a maximal ideal that is principal, $\mathfrak{O}/\mathfrak{P} \cong GF(q)$, a finite field. There is a norm on K, $|\cdot|_K: K^* \rightarrow [0, \infty)$, such that $|x + y|_K \leqq \max[|x|_K, |y|_K]$. (This is known as the ultrametric inequality.) $\mathfrak{O} = \{|x|_K \leqq 1\}$. $\mathfrak{P} = \{|x|_K < 1\}$.