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FORMULAS FOR THE NEXT PRIME

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In 1971, J. M. Gandhi showed that if the first n primes, $p_1, p_2, \cdots, p_n$ are known, then the next prime, p_{n+1}, is given "explicitly" by the formula:

$$(1) \qquad 1 < b^t\Big(\sum_{d|P_n} \frac{\mu(d)}{b^d - 1} - \frac{1}{b}\Big) < b\,,$$

where b is any positive integer $\geqq 2$, where $P_n = p_1p_2 \cdots p_n$, where $\mu(d)$ is the Möbius function, and where the unique integer value of t which satisfies the indicated inequalities is in fact p_{n+1}.

In this paper, we obtain of the following formulas for p_{n+1}:

$$(2) \qquad p_{n+1} = \lim_{s\to\infty} \{P_n(s)\zeta(s) - 1\}^{-1/s}$$

$$(3) \qquad p_{n+1} = \lim_{s\to\infty} \{P_n(s) - \zeta^{-1}(s)\}^{-1/s}$$

$$(4) \qquad p_{n+1} = \lim_{s\to\infty} \{\zeta(s) - Q_n(s)\}^{-1/s}$$

and

$$(5) \qquad p_{n+1} = \lim_{s\to\infty} \{1 - \zeta^{-1}(s)Q_n(s)\}^{-1/s}\,.$$

Here $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for (real) $s > 1$ is the Riemann Zeta Function, with $\zeta^{-1}(s) = \sum_{n=1}^{\infty} \mu(n)/n^s$; $P_n(s) = \prod_{p_i|P_n}(1 - p_i^{-s})$, and $Q_n(s) = \{P_n(s)\}^{-1} = \sum_{n=1}^{\prime\infty} n^{-s}$, where the *prime* indicates that summation is extended over those values of n having no prime factors exceeding p_n.

The approach to be followed here involves the derivation of a more general formula, based on the notion of probability distributions on the positive integers, from which both the Gandhi formula and the new formulas listed above follow as special cases.

2. Probability formulas for the integers. Let $\alpha(n)$ be a *probability function* on the positive integers. That is, $\alpha(n) \geqq 0$ for all $n = 1, 2, 3, \cdots$, and $\sum_{n=1}^{\infty} \alpha(n) = 1$.

Let $\beta(m) = \sum_{n=1}^{\infty} \alpha(mn)$. In the probability distribution D determined by $\{\alpha(n)\}$, $\beta(m)$ is the *probability that a randomly chosen integer is a multiple of m*. Next, let $\gamma(k) = \sum_{d|k} \mu(d)\beta(d)$. Then $\gamma(k)$ is the probability (in D) that a randomly chosen integer is relatively prime to k, because

$$\gamma(k) = 1 - \sum_{p_i|k} \beta(p_i) + \sum_{p_ip_j|k} \beta(p_ip_j) - + \cdots\,.$$

Let $P_n = p_1p_2 \cdots p_n$ be the product of the first n primes. Then