PACIFIC JOURNAL OF MATHEMATICS
Vol. 63, No. 2, 1976

EXACT FUNCTORS AND MEASURABLE CARDINALS

ANDREAS BLASS

The purpose of this paper is to prove that all exact functors from the category $\mathscr{S}$ of sets to itself are naturally isomorphic to the identity if and only if there are no measurable cardinals. The first step in the proof is to approximate arbitrary left-exact endofunctors F of $\mathscr{S}$ with endofunctors of a special sort, reduced powers, and to characterize reduced powers in terms of category-theoretic properties. The next step is to determine the effect, on the approximating reduced powers, of the additional assumption that F preserves coproducts or coequalizers. It turns out that preservation of coequalizers is an extremely strong condition implying preservation of many infinite coproducts. From this fact, the main theorem follows easily.

Left-exact functors, by definition, preserve equalizers and finite products. It follows that they preserve pullbacks (including intersections as a special case) and monomorphisms. Note that any functor on $\mathscr{S}$ preserves all epimorphisms, because they split. Also note that, if $F: \mathscr{S} \to \mathscr{S}$ is left exact and $F(\phi) = \phi$, then, for F to preserve a coproduct $\coprod_{\alpha \in I} A_\alpha$ with injections i_α, it is necessary and sufficient that the maps $F(i_\alpha)$ be jointly epic; indeed, left-exactness guarantees that these maps are monic and that the ranges of any two of them have intersection $F(\phi)$ which was assumed to be empty.

To avoid annoying special cases later, observe that there is only one (up to natural isomorphism) product-preserving $F: \mathscr{S} \to \mathscr{S}$ for which $F(\phi) \neq \phi$, namely the functor sending every set to a singleton (i.e. the functor represented by ϕ). To see this, simply note that the second projection $F(X) \times F(\phi) \to F(\phi)$ is an isomorphism (because $X \times \phi \to \phi$ is an isomorphism and F preserves products). This functor will be called the *improper* left-exact endofunctor of $\mathscr{S}$; all others are *proper*.

For the sake of notational simplicity, when F is a product-preserving functor, the natural isomorphism $F(A \times B) \cong F(A) \times F(B)$ (induced by F of the projections) will not be explicitly mentioned. Thus, if $a \in F(A)$ and $b \in F(B)$, then (a, b) will be considered an element of $F(A \times B)$.

For similar reasons, the distinction between sets, classes, and things of even higher type will be suppressed (except in § 5). For example, the category of left-exact endofunctors of $\mathscr{S}$ and natural transformations between them will be treated as though it were a set. Scrupulous readers are invited to assume the existence of a