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ON THE INVERSE FUNCTION THEOREM

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The classical inverse function theorem gives conditions under which a C^r function admits (locally) a C^r inverse. The purpose of this article is to give conditions under which a Lipschitzian (not necessarily differentiable) function admits (locally) a Lipschitzian inverse. The classical result is a special case of the theorem.

1. Introduction. Let $f\colon R^n \to R^n$ satisfy a Lipschitz condition in a neighborhood of a point x_0 in R^n. Thus for some constant K, for all x and y near x_0, we have

$$|f(x)-f(y)| \leqq K\,|x-y|, \tag{1}$$

where $|\cdot|$ denotes the usual Euclidean norm. The usual $n \times n$ Jacobian matrix of partial derivatives, when it exists, is denoted $Jf(x)$. We topologize the vector space $\mathcal{M}$ of $n \times n$ matrices with the norm

$$\|M\| = \max |m_{ij}|,$$

where

$$M = (m_{ij}), \qquad 1 \leqq i \leqq n, \qquad 1 \leqq j \leqq n.$$

DEFINITION 1. The *generalized Jacobian* of f at x_0, denoted $\partial f(x_0)$, is the convex hull of all matrices M of the form

$$M = \lim_{i \to \infty} Jf(x_i),$$

where x_i converges to x_0 and f is differentiable at x_i for each i.

The above extends to vector-valued functions the notion of "generalized gradient" introduced by the author in **[2]**. It is a consequence of Rademacher's theorem that f is almost everywhere differentiable near x_0. Furthermore, $Jf(x)$ is bounded near x_0 as a result of (1). These observations imply

PROPOSITION 1. *$\partial f(x_0)$ is a nonempty compact convex subset of $\mathcal{M}$.*

DEFINITION 2. $\partial f(x_0)$ is said to be of *maximal rank* if every M in $\partial f(x_0)$ is of maximal rank.

The following theorem, which is our main result, is proven in §2: