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ANOTHER MARTINGALE CONVERGENCE THEOREM

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A classical martingale theorem is generalized to "martingale like" sequences. The method of proof is a generalization of Doob's proof by "downcrossings".

Introduction. Let (Ω, B, P) be a probability space, $\{B_n\}$ an increasing sequence of sub sigma fields of B. Let $\{f_n, B_n, n \geqq 1\}$ be an adapted sequence of P-integrable random variables.

The sequence is said to be a *martingale in the limit* if

$$\lim_{n\to\infty} \sup_{\bar{n}>n} |f_n - E(f_{\bar{n}} \mid B_n)| = 0 \qquad P \quad \text{a.e.}$$

If was proven in an earlier paper, Mucci [3] that every uniformly integrable martingale in the limit converges both L_1 and P a.e., generalizing the corresponding martingale theorem. The purpose of the present note is to prove that every L_1-bounded martingale in the limit converges pointwise to an integrable random variable, thereby generalizing another classical martingale theorem. We recall that a sequence $\{f_n\}$ is said to be L_1-bounded if $\sup_n \int |f_n| < \infty$.

THE THEOREM. *Let $\{f_n, B_n, n \geqq 1\}$ be an L_1-bounded martingale in the limit. Then there exists $f \in L_1$ with $f_n \to f$ P a.e.*

Proof. Fix $a < b$, two arbitrary real numbers. We define, following the classical proof:

$\varphi(a, b)$ is the number of "downcrossings" of $\{f_n\}$ from above b to below a. Our objective will be to show that $P(\varphi(a, b) = \infty) = 0$ so that $P(\underline{\lim}\, f_n \leqq a < b \leqq \overline{\lim}\, f_n) = 0$, thereby determining that $f = \lim_n f_n$ exists almost everywhere, and since

$$\int |f| < \underline{\lim} \int |f_n| < \infty; \quad \text{that} \quad f \in L_1.$$

Our procedure consists in defining a "modified" number of downcrossings $\bar{\varphi}(a, b)$ and showing that $P(\bar{\varphi}(a, b) = \infty) = 0$ and further that, almost everywhere,

$$\bar{\varphi}(a, b) < \infty \quad \text{implies} \quad \varphi(a, b) < \infty.$$