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# PARTIAL REGULARITY OF SOLUTIONS TO THE NAVIER–STOKES EQUATIONS

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**At the first instant of time when a viscous incompressible fluid flow with finite kinetic energy in three space becomes singular, the singularities in space are concentrated on a closed set whose one dimensional Hausdorff measure is finite.**

**§1. Introduction.** Let $v: R^3 \times R^+ \to R^3$ (where $R^+ = \{t \in R: t > 0\}$ represents time) be a weak solution to the Navier–Stokes equations of incompressible viscous fluid flow in 3 dimensional euclidean space with finite initial kinetic energy and viscosity equal to 1. Our definition of weak solution coincides with Leray's definition of "solution turbulente" [**4**, pp. 240, 241, 235]. In that paper, Leray showed that weak solutions always exist for prescribed initial conditions with finite energy. He also proved the following regularity theorem:

LERAY'S THEOREM. *There exists a finite or countable sequence $J_0$, $J_1$, $J_2, \cdots$ such that $J_q \subset R^+$, $J_0 = \{t: t > a\}$ for some $a$, $J_q$ is an open interval for $q > 0$, the $J_q$ are disjointed, the Lebesgue measure of $R^+ - \bigcup_{q \geqq 0} J_q$ is zero, $v$ can be modified on a set of Lebesgue measure zero so that its restriction to each $R^3 \times J_q$ becomes smooth, and*

$$\sum_{q>0} (\text{length}\,(J_q))^{1/2}$$

*is finite.*

It is not known whether there exist $v$ with singularities ($J_0 = R^+$ is a possibility). The purpose of this paper is to prove the following theorem on the nature of possible singularities of $v$. We assume that $v$ has been modified to be smooth on each $R^3 \times J_q$.

THEOREM 1. *Let $t_0$ be the right endpoint of an interval $J_q$ with $q > 0$. Then there exists a closed set $S \subset R^3$ such that $v$ can be extended to a continuous function on*

$$(R^3 \times J_q) \cup ((R^3 - S) \times \{t_0\})$$

*and the 1 dimensional Hausdorff measure of $S$ is finite.*