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SOME FORMS OF ODD DEGREE FOR WHICH THE HASSE PRINCIPLE FAILS

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The object of this paper is to give a family of absolutely irreducible forms of odd degree for which the Hasse principle fails.

Let K be an algebraic number field and $f(X_1, \cdots, X_n)$ be a polynomial of n variables $X_1, \cdots, X_n$ over K. We say the Hasse principle over K, briefly H.P./K, holds for $f(X_1, \cdots, X_n)$ when $f(X_1, \cdots, X_n) = 0$ has a solution in K if and only if it has a solution in $K_\mathfrak{p}$ for all prime spots $\mathfrak{p}$. Here, if f is a form, a solution means a nontrivial one. Our aim in this paper is to push forward the method in [3] and thus producing a family of forms of odd degree for which H. P. fails. As is well known, the Hasse-Minkowski theorem assures the validity of H. P. for any quadratic forms. So far as forms of higher degree are concerned, the things are not so simple if the form is absolutely irreducible of odd degree (see [1], Chap. I, § 7). For forms of degree 3, there have been found several counter examples ([2], [4], [5], [6]). Such a form of degree 5 was discovered by the first author [3]. In this paper, we prove the following theorem. Let $\boldsymbol{P}$ be the set of primes which satisfy the conditions in § 2 of this paper. For example $\{p \in \boldsymbol{P};\ p \leqq 1000\} = \{17, 53, 89, 131, 149, 167, 179, 257, 311, 359, 431, 449, 467, 521, 563, 599, 683, 773, 887, 953, 977\}$.

THEOREM. H.P./$\boldsymbol{Q}$ *does not hold for the following form of degree* $10n + 5$

$F(x, y, z) = (x^3 + 5y^3)(x^2 + xy + y^2)^{5n+1} - pz^{10n+5}$, *where* n *is any nonnegative integer and* p *is in* $\boldsymbol{P}$.

This theorem gives counter examples, for H. P./$\boldsymbol{Q}$, of any odd degree divisible by 5. Though the method of the proof is basically analogous to that of [3], local solvability needs more careful and involved treatment.

In § 1 we prove that the equation $F = 0$, actually in a slightly more general setting, can be solved everywhere locally. It goes without saying that Hensel's lemma plays a central role there.

In § 2 we show that the equation does not have any integral, therefore rational, solution. The argument used there enables one to find as many primes in $\boldsymbol{P}$ as one may want. We remark here