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INVARIANT MEASURES FOR ERGODIC SEMIGROUPS OF OPERATORS

RYOTARO SATO

In this paper the invariant measure problem is investigated for ergodic semigroups (in the sense of Eberlein) of positive linear operators on the L_1-space of a probability measure space. Various results in operator ergodic theory are generalized and extended to give a unified approach to the problem. The main step in this approach is the following result: There exists a positive linear functional φ on the space $B(\Lambda)$ of all bounded real valued functions on a directed set Λ such that

$$\liminf_{\alpha \in \Lambda} \xi(\alpha) \leqq \varphi(\xi) \leqq \limsup_{\alpha \in \Lambda} \xi(\alpha)$$

for all $\xi \in B(\Lambda)$.

Let $(X, \mathfrak{M}, m)$ be a probability measure space and let $L_p(X) = L_p(X, \mathfrak{M}, m)$, $1 \leqq p \leqq \infty$, be the Banach spaces defined as usual with respect to $(X, \mathfrak{M}, m)$. For a set $A \in \mathfrak{M}$, 1_A denotes the indicator function of A and $L_p(A)$ denotes the Banach space of all $L_p(X)$-functions that vanish a.e. on $X - A$. If $f \in L_p(X)$, we define supp f to be the set of all x in X at which $f(x) \neq 0$. Relations introduced below are assumed to hold modulo sets of m-measure zero.

Let $\Sigma = \{T\}$ be a semigroup of positive linear operators on $L_1(X)$. A function $f \in L_1(X)$ is called *Σ-fixed* if $Tf = f$ for every $T \in \Sigma$. The problem of finding necessary and sufficient conditions for the existence of a Σ-fixed $f_0 \in L_1(X)$, with $f_0 > 0$ a.e. on X, has been studied by many authors (see, for example, [4], [5], [8], [9], [11], [12], [13], [14], [17], [18], [21], [22], [23], [24], [25], [27], and others). In the present paper we intend to investigate the problem for ergodic semigroups Σ in the sense of Eberlein, and generalize and extend various known results to give a unified approach to the problem.

For $f \in L_1(X)$, we denote by $\overline{\text{co}}\, \Sigma f$ the closed convex hull of the set $\{Tf : T \in \Sigma\}$. Σ is said to be *left* [resp. *right*] *ergodic* if there exists a net $(T_\alpha, \alpha \in \Lambda)$ of positive linear operators on $L_1(X)$ satisfying

(a) $\limsup_\alpha \|T_\alpha\| < \infty$,
(b) for every $f \in L_1(X)$ and every $\alpha \in \Lambda$, $T_\alpha f \in \overline{\text{co}}\, \Sigma f$,
(c) for every $T \in \Sigma$, $\lim_\alpha TT_\alpha - T_\alpha T = 0$ [resp. $\lim_\alpha T_\alpha - T_\alpha = 0$],

where the convergence can be either in the uniform, strong, or weak operator topology. (Cf. Eberlein [7] and Day [3].) The above net $(T_\alpha, \alpha \in \Lambda)$ is said to be *left* [resp. *right*] *Σ-ergodic*. If $(T_\alpha, \alpha \in \Lambda)$ is