PACIFIC JOURNAL OF MATHEMATICS
Vol. 71, No. 2, 1977

A NOTE ON THE GROUP STRUCTURE OF UNIT REGULAR RING ELEMENTS

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Local properties of unit regular ring elements are investigated. It is shown that an element of a ring R with unity is regular if and only if there exists a unit $u \in R$ and a group G such that $a \in uG$.

1. Introduction. It is well-known that [15, 7] a ring R is strongly regular if and only if every $a \in R$ is a group member. In this note we shall use the basic theorem for group members in a ring to show locally that a ring element $a \in R$ (with unity) is unit regular exactly when there is a unit $u \in R$ and a group G in R such that $a \in uG$. Hence unit regular rings are, as it were locally a "rotated" version of strongly regular rings.

We remind the reader that a ring R is called regular if for every $a \in R$, $a \in aRa$; strongly regular if for every $a \in R$, $a \in a^2R$, and unit regular if for every $a \in R$, there is a unit $u \in R$ such that $aua = a$ [3]. Similar definitions hold locally. A ring with unity is called finite if $ab = 1$ implies $ba = 1$. Any solution a^- to $axa = a$ is called an inner or 1-inverse of [1], while any solution a^+ to $axa = a$ and $xax = x$ is called a reflexive or 1-2 inverse of a.

For idempotents e and f in R, $e \sim f$ denotes the equivalence in the sense of Kaplansky [13] as contrasted with $a \overset{u}{\sim} b$ which denotes that $a = pbq$ with p and q invertible.

As usual, similarity will be denoted by $\approx$, the right and left annihilators of $a \in R$ will be denoted by $a^0 = \{x \in R: ax = 0\}$, ${}^0a = \{x \in R: xa = 0\}$ respectively, while interior direct sums and isomorphisms are denoted by $\dotplus$ and $\cong$ respectively. A ring R is called faithful if $aR = (0)$ implies $a = 0$.

We shall make use of the following fundamental theorem for group members.

THEOREM 1. *Let S be a semigroup and $a \in S$. The following are equivalent.*

1. *a is a group member.*
2. *a has a group inverse $a^\#$ in S which satisfies $axa = a$, $xax = x$ and $ax = xa$.*
3. *a has a commutative inner inverse a^- which satisfies $axa = a$, and $ax = xa$.*
4. *$aS = eS$, $Sa = Se$ and $a \in eSe$ for some idempotent $e \in S$.*
5. *$a \in a^2S \cap Sa^2$.*