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ON INTEGRAL REPRESENTATIONS OF PIECEWISE HOLOMORPHIC FUNCTIONS

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Let D be the interior of the unit circle in C, D^c its exterior and T the unit circumference. We consider certain piecewise holomorphic functions that are holomorphic in D and also in D^c. This paper deals with those piecewise holomorphic functions that are representable by means of complex Poisson-Stieltjes integrals on T; we call this set of functions P_1. The set of all piecewise holomorphic functions (holomorphic in D and in D^c) we call P. Earlier work—see Rolf Nevanlinna, Eindeutige Analytische Funktionen, Springer, Berlin, 1953 and references there—dealt with positive (Herglotz-Riesz) or real (Nevanlinna) measures; we shall use here the entire space M of bounded complex Borel measures on T. This gives the theory more flexibility. We consider characterizations of functions in P representable by means of complex Poisson-Stieltjes integrals, uniqueness questions, the nature of the mapping between the subset P_1 of P of representable functions and M, as well as the ring structures in M (under convolution) and P_1 (Hadamard products), and questions of derivatives and integrals. We end with an application to Fourier-Stieltjes moments relative to measues in M.

We call a function $F \in P$ *representable* if there is a measure $m \in M$ so that $F = \int P_C dm + k$ where $P_C = P_C(z) = (e^{it} + z)/(e^{it} - z)$ is the complex Poisson kernel, k is a piecewise constant function in P, and where the limits of integration are omitted when they are 0 and 2π respectively. A function $F \in P$ is said to be of *real type* if $F(\bar{z}^{-1}) = -\overline{F(z)}$ for all $z \in D \cup D^c$. The functions

$$(1) \quad G = G_F(z) = \frac{1}{2}(F(z) - \overline{F(\bar{z}^{-1})}),\ H = H_F(z) = -\frac{1}{2}(iF(z) + i\overline{F(\bar{z}^{-1})})$$

are of real type; we have $F = G + iH$ and $F \in P_1$ if and only if G and H are in P_1.—The decomposition of the complex measure m into its real and imaginary parts is given by $m = (1/2(m + \bar{m})) + i((1/2i)(m - \bar{m})) = (\operatorname{Re} m) + i(\operatorname{Im} m)$ where $\bar{m}$ is defined as usual by $\int \bar{g}\, dm = \overline{\int g\, dm}$ for continuous functions g on T. If the representable function $F \in P_1$ is given by $F = \int P_C dm + k$, then $G_F = \int P_C d(\operatorname{Re} m) + 1/2(k - \bar{k})$ and $H_F = \int P_C d(\operatorname{Im} m) + (1/2i)(k + \bar{k})$. — If $m \in M$, we write $\hat{m}_j = \int e^{-ijt} dm$.