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GENERATING $O(n)$ WITH REFLECTIONS

MORRIS L. EATON AND MICHAEL PERLMAN

For $r \in C_n \equiv \{x|x \in R^n, ||x|| = 1\}$, let $S_r = I_n - 2rr'$ where r is a column vector. $O(n)$ denotes the orthogonal group on R^n. If $R \subseteqq C_n$, let $\mathscr{R} = \{S_r|r \in R\}$ and let G be the smallest closed subgroup of $O(n)$ which contains $\mathscr{R}$. G is *reducible* if there is a nontrivial subspace $M \subseteqq R^n$ such that $gM \subseteqq M$ for all $g \in G$. Otherwise, G is *irreducible*.

THEOREM. **If G is infinite and irreducible, then $G = O(n)$.**

In what follows, R^n denotes Euclidean n-space with the standard inner product, $O(n)$ is the orthogonal group of R^n, and $C_n = \{x \mid x \in R^n, ||x|| = 1\}$. If U is a subset of $O(n)$, $\langle U\rangle$ denotes the group generated algebraically by U and $\langle \bar{U}\rangle$ denotes the closure of $\langle U\rangle$. Thus, $\langle \bar{U}\rangle$ is the smallest closed subgroup of $O(n)$ containing U. For an integer k, $1 \leqq k < n$, M_k denotes a k-dimensional linear subspace of R^n. If $r \in C_n$, let $S_r = I - 2rr'$ where r is a column vector. Thus S_r is a reflection through r-henceforth called a reflection.

Suppose $R \subseteqq C_n$ and let $\mathscr{R} = \{S_r \mid r \in R\}$. Set $G = \langle \bar{\mathscr{R}}\rangle$. The group G is *reducible* if there is an M_k such that $gM_k \subseteqq M_k$ for all $g \in G$; otherwise, G is *irreducible*. The main result of this note is the following.

THEOREM 1. *If G is infinite and irreducible, then $G = O(n)$.*

Proof of Theorem 1. First note that if $S_r \in \mathscr{R}$ and $g \in G$, then $gS_rg^{-1} = S_{gr} \in G$. Let $\Delta = \{gr \mid g \in G, r \in R\}$. Thus, $t \in \Delta$ implies that $S_t \in G$. Since G is infinite, Δ must be infinite (see Benson and Grove (1971), Proposition 4.1.3). Since every Γ in $O(n)$ is a product of a finite number of reflections, to show that $G = O(n)$, it suffices to show that G is transitive on C_n (if G is transitive on C_n, then $\Delta = C_n$ so every reflection is an element of G and hence $G = O(n)$).

The proof that G is transitive on C_n follows. By Lemma 1 (below), there is a subgroup $K_2 \subseteqq G$ and a subspace $M_2 \subseteqq R^n$ such that $kx = x$ if $x \in M_2^{\perp}$ and $k \in K_2$ and K_2 is transitive on $D_2 \equiv M_2 \cap C_n$. Since G is irreducible, there is an $r_2 \in R$ such that $r_2 \notin M_2$ and $r_2 \notin M_2^{\perp}$. Let $M_3 = \text{span}\{r_2, M_2\}$ and let $K_3 = \langle\{K_2, S_{r_2}\}\rangle > \subseteqq G$. With $D_3 \equiv M_3 \cap C_n$, Lemma 3 (below) implies that $kx = x$ for all $x \in M_3^{\perp}$ and $k \in K_3$, and K_3 is transitive on D_3. Again, since G is irreducible, there is an $r_3 \in R$ such that $r_3 \notin M_3$ and $r_3 \notin M_3^{\perp}$. With $M_4 = \text{span}\{r_3, M_3\}$, let $K_4 = \langle\{K_3, S_{r_3}\}\rangle > \subseteqq G$ and let $D_4 \equiv M_4 \cap C_n$. By Lemma 3 (below)