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ON COVERINGS OF EUCLIDEAN SPACE BY CONVEX SETS

G. D. CHAKERIAN AND H. GROEMER

Let $\mathscr{K} = \{K_1, K_2, \cdots\}$ be an infinite countable class of compact convex subsets of euclidean n-dimensional space R^n. We shall say that $\mathscr{K}$ permits a space covering or, more precisely, a covering of R^n, if there are rigid motions $\sigma_1, \sigma_2, \cdots$ such that $R^n \subset \bigcup_{i=1}^{\infty} \sigma_i K_i$. In this paper we concern ourselves with necessary and sufficient conditions in order that a given class $\mathscr{K}$ permits a space covering.

If the set of diameters $\{d(K_i): K_i \in K\}$ is bounded, the problem has already been solved in **[3]** by showing that in this case $\mathscr{K}$ permits a covering of R^n if and only if the series $\Sigma_{i=1}^{\infty} v(K_i)$ of volumes $v(K_i)$ diverges. (The same result holds obviously without any restrictions on the diameters if $n = 1$.) On the other hand, if $\{d(K_i)\}$ is unbounded and $n > 1$, it is not difficult to see (cf. **[1]** and **[2]**) that the divergence of this series is no longer sufficient but only necessary. Only in the special case $n = 2$ are some necessary and sufficient conditions known **[2]**.

Our principal results are stated in the following §2. Theorem 1 gives an inductive criterion that enables one to decide whether a given $\mathscr{K}$ permits a space covering. Theorems 2 and 3 serve the same purpose but are of a more explicit nature, involving the divergence of infinite series of geometric invariants associated with the members of $\mathscr{K}$. Other results, regarding coverings by balls, boxes (i.e. isometric images of n-dimensional intervals), and 2-dimensional sets, are stated and discussed in the same section. This is followed by the proofs of three lemmas in §3. Lemma 1 appears to be of some independent interest. §4 contains the proofs of our theorems.

2. Theorems and corollaries.

A nonempty compact convex set will be called a convex body. If K is a convex body in R^n, and if p, q are two points of K such that the length of the segment $[p, q]$ is equal to the diameter $d(K)$, then we call the orthogonal projection of K onto a hyperplane perpendicular to $[p, q]$ a normal projection of K. Of course, a normal projection of K is not uniquely determined by K. However, if $\{K_i\}$ is given we shall always assume that for each K_i a definite normal projection $N(K_i)$ has been selected and is kept fixed. Since each $N(K_i)$ is at most $(n-1)$-dimensional it is clear (using a self-explanatory extension of our original definition) what is meant by saying that the class