PACIFIC JOURNAL OF MATHEMATICS
Vol. 76, No. 2, 1978

RINGS WITH QUIVERS THAT ARE TREES

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Associated with each artinian ring R are two diagrams called the left and right quivers of R. We generalize a well-known theorem on hereditary serial rings by proving that if these quivers have no closed paths then R is a factor ring of a certain ring of matrices over a division ring. It follows that the categories of finitely generated left and right R-modules are Morita dual to one another. Applying our theorem and theorems of Gabriel and Dlab and Ringel, we show how to write explicit matrix representations of all hereditary algebras of finite module type.

A *quiver* is, in the terminology of Gabriel [8], [9], a finite set of points (vertices) connected by arrows. Given an artinian ring R and a basic set of primitive idempotents $e_1, \cdots, e_n$ of R (see, for example, [1, §27]), one forms $\mathscr{Q}({}_RR)$ the *left quiver of* R (see [11]): The vertices of $\mathscr{Q}({}_RR)$ are $v_1, \cdots, v_n$, one for each basic idempotent, with n_{ij} arrows from v_i to v_j iff Re_j/Je_j appears exactly n_{ij} times in a direct decomposition of the semisimple left R-module Je_i/J^2e_i. The right quiver $\mathscr{Q}(R_R)$ is formed similarly, with vertices $v'_1, \cdots, v'_n$ and n'_{ij} arrows from v'_i to v'_j iff e_jR/e_jJ appears exactly n'_{ij} times in a direct decomposition of e_iJ/e_iJ^2. Note that $n'_{ij} \neq 0$ iff $n_{ji} \neq 0$. Also, R is indecomposable iff $\mathscr{Q}({}_RR)$ is connected, i.e., there is a nonoriented path from v_i to v_j for every $i, j = 1, \cdots, n$.

A quiver $\mathscr{Q}$ is called a *tree* in case it is connected and contains no cycles, i.e., in case it has a unique nonoriented path from v_i to v_j, for every i, j. Let $\mathscr{Q}$ be such a quiver. Then the vertices of $\mathscr{Q}$ are partially ordered by $\leqq$, where $v_i \leqq v_j$ iff there is an oriented path from v_j to v_i (or $i = j$), and we can relabel the vertices so that $v_i \nleqq v_j$ implies $i \leqq j$. Having done this, we see that for any ring D, the set of matrices

$$T = \{[\![d_{ij}]\!] \mid d_{ij} \in D,\ d_{ij} = 0 \quad \text{if} \quad v_i \nleqq v_j\}$$

is a subring of the ring of upper triangular matrices over D. Moreever, if D is a division ring, then $\mathscr{Q}({}_TT) = \mathscr{Q}$, $\mathscr{Q}(T_T)$ is the dual quiver of $\mathscr{Q}$, and T is the unique basic *tic tac toe ring* (in the sense of Mitchell [12, §10.8]) over D with left quiver $\mathscr{Q}$.

Murase [14] showed that an indecomposable artinian ring whose quivers are of the form

$$v_1 \longleftarrow v_2 \longleftarrow v_3 \cdots v_{n-1} \longleftarrow v_n$$