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PROBABILITY MEASURES AND THE C-SETS OF SELIVANOVSKIJ

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Let X be a Borel space and $\mathscr{S}(X)$ the smallest σ-field containing the Borel subsets of X and closed under operation (A). Then $\mathscr{S}(X)$ is a sub-σ-field of the class of universally measurable subsets of X. Let $P(X)$ be the space of probability measures on the Borel subsets of X and equip $P(X)$ with the weak topology. It is proved that if $A \in \mathscr{S}(X)$, then $\{p \in P(X): P(A) \geqq \lambda\}$ is in $\mathscr{S}[P(X)]$ for every real λ. This allows us to shows that if X and Y are Borel spaces, f is an $\mathscr{S}(X \times Y)$ measurable extended-real-valued function, and $x \rightarrow q(\cdot \mid x)$ is $\mathscr{S}(X)$ measurable from X to $P(Y)$, then $x \rightarrow \int f(x, y) q(dy \mid x)$ is $\mathscr{S}(X)$ measurable on the set where it is defined. (Measurability is relative to the Borel σ-fields in the range spaces.)

1. **Introduction.** In the study of sequential optimization, it is often convenient to pose problems in Polish spaces or Borel subsets of Polish spaces (hereafter called *Borel spaces*). Such a space together with its Borel subsets is a well behaved measurable space, but there are certain simple operations which lead one out of the Borel sets into larger σ-fields. One of these is infimization: if $f: X \times Y \rightarrow R^*$ is Borel, where X and Y are Borel spaces and $R^* = R \cup \{\pm \infty\}$ is a Borel space with the usual topology, then

$$g(x) = \inf_{y \in Y} f(x, y) \tag{1.1}$$

may fail to be Borel, but will be analytically measurable, i.e., measurable with respect to the σ-field generated by the analytic sets in X [10]. Another such operation is selection: if $B \subset X \times Y$ is Borel and the projection of B onto X is X, there may fail to exist a Borel measurable function $\phi: X \rightarrow Y$ whose graph lies in B [2], but by a famous result due independently to Jankov [6] and von Neumann [11], a selector ϕ which is analytically measurable can always be found. (See also [8] and [3].) Another selection problem is this. Let g be as in (1.1) and $\varepsilon > 0$. Does there exist a measurable $\phi: X \rightarrow Y$ such that

$$f[x, \phi(x)] \leqq \begin{cases} g(x) + \varepsilon & \text{if} \quad g(x) > -\infty \\ -\dfrac{1}{\varepsilon} & \text{if} \quad g(x) = -\infty ? \end{cases} \tag{1.1}$$

Such a selector which is Borel measurable need not exist, but one