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THE CASE OF EQUALITY IN THE MATRIX-VALUED TRIANGLE INEQUALITY

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This paper presents an analysis of the case of equality in the matrix-valued triangle inequality. There is complete analogy with the case of equality in the usual scalar triangle inequality.

In order to describe our assertion more precisely, let A and B be n-square complex matrices, and by $|A|$ denote the positive semidefinite Hermitian matrix

$$|A| = (AA^*)^{1/2} ,$$

where A^* is the adjoint of A. It has been speculated several times in the literature that this inequality should "naturally" hold:

$$|A + B| \leqq |A| + |B| ,$$

where the inequality sign signifies that the right hand side minus the left hand side is positive semidefinite. This inequality is false, however, as easy 2×2 examples show. Nevertheless, there is a valid matrix valued triangle inequality. It was discovered in [1], and takes the form

$$|A + B| \leqq U|A|U^* + V|B|V^* \tag{1}$$

for appropriately chosen unitary matrices U and V (dependent upon A and B). However, no analysis of a "case of equality" for (1) was given in [1], and the purpose of this note is to supply such an analysis. Specifically, we have:

THEOREM 1. *The inequality sign in* (1) *must be equality if A and B have polar decompositions with a common unitary factor.*

THEOREM 2. *Suppose A and B are such that inequality* (1) *can hold only with the equality sign. Then A and B have polar factorizations with a common unitary factor.*

Proof of Theorem 1. We have $A = WH$ and $B = WK$, where W is unitary and H, K are positive semidefinite Hermitian. From (1) we easily deduce that

$$H + K \leqq U_1HU_1^* + V_1KV_1^* ,$$