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T AS AN $\mathscr{E}$ SUBMODULE OF G

W. J. WICKLESS

Let G be a mixed abelian group with torsion subgroup T. T is viewed as an $\mathscr{E}$ submodule of G, where $\mathscr{E} = \mathrm{End}\, G$. It is shown that T is superfluous in G if and only if, $\forall_p$, either T_p is divisible or G/T_p is not p divisible. If G is not reduced, T is essential in G if and only if T contains a $Z(p^\infty)$. Let $I(G)$ $[I(T)]$ be the $\mathscr{E}$ injective hull of G $[T]$. Then $I(G) = I(T) \oplus X$ with X torsion free divisible and T is a pure subgroup of $I(G)$. This can be used to obtain several results; for example, if $Q \not\subseteq I(T)$, TFAE: 1. $T \,\mathrm{ess}\, G$, 2. $I(G) \cong I(T)$ as abelian groups, 3. $Q \not\subseteq I(G)$. The condition $T \,\mathrm{ess}\, G$ is characterized if T is a summand or if G is algebraically compact. If T is bounded or if T is a p-group, $T^1 = (0)$ and G is reduced cotorsion, T is not essential. In fact, for bounded T there is an $\mathscr{E}$ isomorphism $I(G) \cong I(T) \oplus I(G/T)$. Some information is obtained on the p-basic subgroups of $I(T)$ as a function of those of T. A condition is given for $I(T) \supseteq \oplus_c Q$. These last theorems specialize to $I({}_E T)$, where $E = \mathrm{End}\, T$.

Preliminaries. In the last fifteen years several authors have written papers concerning an abelian group G viewed as a module over $\mathscr{E}$, its ring of endomorphisms.

Let G be a mixed abelian group with maximal torsion subgroup T. In this paper we consider T as an $\mathscr{E}$ submodule of G. We determine when T is superfluous in G and then study the more difficult question of determining when T is essential in G. (If $(0) \neq T \neq G$, it is easy to prove that T is neither essential nor superfluous as a Z submodule of G.)

The latter question leads to consideration of the injective hulls $I(T)$, $I(G)$—taken with respect to $\mathscr{E}$.

Our notation, with minor exceptions, is that of [1].

1. T as a superfluous submodule of G.

Henceforth, let G be a mixed abelian group, $T = t(G)$ its torsion subgroup and $\mathscr{E} = \mathrm{End}\, G$. To avoid stating the trivial cases of our results we always assume $(0) \neq T \neq G$. We begin by characterizing those mixed G for which ${}_{\mathscr{E}}T$ is superfluous in ${}_{\mathscr{E}}G$ $(T \ll G)$. In our context $T \ll G$ if and only if whenever K is a fully invariant subgroup of G with $K + T = G$, then $K = G$.

LEMMA 1. *Let $T = \bigoplus T_p$ be a decomposition of T into its p components. Then $T \ll G$ if and only if $T_p \ll G$, $\forall p$.*