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METRIC AVERAGING IN EUCLIDEAN AND HILBERT SPACES

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A number of geometric properties of sets in l^2 can be measured in terms of maxima and minima of quadratic forms subject to side conditions. A method of metric addition which exploits this fact is investigated. The main objective is to develop a flexible method for attacking geometric extremal problems involving sums of distances or other metric constraints.

1. Introduction and principal embedding theorem. The general approach for metric addition, or averaging, in Euclidean and Hilbert spaces has proved useful in the study of a number of geometric extremal problems (See [1], [2].) However, in this earlier work we did not realize that the various averaging processes and inequalities could be presented in a completely unified manner.

Obviously, we can make no claim of total originality since special cases of our results are well-known. In fact in §6 we point out that an inequality is equivalent to a known result for positive definite matrices. However we do feel that our method, based on ideas related to an embedding Theorem of I. J. Schoenberg [9], possesses scope and flexibility.

The general method exploits the fact that a number of geometric properties of sets in l^2 can be measured in terms of maxima and minima of quadratic forms subject to linear side conditions.

The logic of the method's application to extremal problems parallels that of Minkowski (or vector) addition. One assumes that an extremal configuration is at hand, and by "adding" suitable copies of the configuration one hopes to determine the nature of the configuration by use of inequalities and other properties which are preserved by the addition process. In §7 this is illustrated by an example.

Let (Y, τ) be a compact topological space which can be homeomorphically embedded in the classical Hilbert space l^2, and let T be a family of homeomorphisms from Y into l^2 such that the images Y_t, t in T, are uniformly bounded. Furthermore we assume the existence of a suitable probability measure μ on subsets of T. The nature of the measure μ is clear in the context of a problem, and it is generally chosen to be a "uniform" measure on T. Define the function d on $Y \times Y$ by the formula