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INJECTIVE HULLS OF GROUP RINGS

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We are concerned in this paper with the following question: When is the maximal right quotient ring of the group algebra kG a right self-injective ring? In general, the maximal right quotient ring $Q(R)$ of a ring R is a right R-submodule of the right injective hull $E(R)$ of R, and we may rephrase our question as: When does $Q(kG) = E(kG)$? Of course, a sufficient condition for this to occur is that kG be right nonsingular, so that, for example, $E(kG) = Q(kG)$ when k is a field of characteristic zero. However, $Q(kG)$ is often injective even when kG is a singular ring; for example, when G is finite, it is well-known that kG is itself an injective ring.

Many of our results are concerned with the case of commutative group algebras. If G is an abelian group and k is a field of positive characteristic p, then $Z(kG)$, the singular ideal of kG, is the ideal generated by the augmentation ideal of the Sylow p-subgroup H of G, [2], Corollary 3.5 and [11], Lemmas 3.1.6 and 8.1.8. We shall prove (Theorem 4.1), that if H contains only finitely many elements of infinite height, then $Q(kG)$ is injective. Not all commutative group algebras have injective quotient rings, however; under the above hypothesis on k, we prove (Corollary 3.2) that if a group T contains a subnormal Prüfer p-group, $Q(kT)$ is not self-injective. Some commutative group algebras are not, of course, included among those covered by Theorem 4.1 and Corollary 3.2, so that even in the commutative case we are unable to offer a complete answer to our question.

Sections 3 and 4 are devoted, respectively, to the proofs of Corollary 3.2, which is in fact a slightly more general result than that quoted above, and Theorem 4.1. The rest of the paper is organized as follows. In §2 we obtain some technical results relating the injective hull of a group algebra to the injective hulls of certain subalgebras and quotient algebras; these results are used frequently in subsequent sections. In §5 we show that, although they fail in general to be injective, maximal quotient rings of commutative group algebras possess many of the well-known properties of injective rings. For example, if Q is such a ring, $Z(Q) = J(Q)$, the Jacobson radical of Q, (Theorem 5.3), $Q/J(Q)$ is a regular self-injective ring, (Theorems 5.3 and 5.5), and idempotents may be lifted over $J(Q)$, (Proposition 5.4). In §6 we drop the assumption, common to §§4 and 5, that the group algebras under consideration are commutative,