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NATURALLY INTEGRABLE FUNCTIONS

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A bounded function f defined on an amenable group G is *naturally integrable* if, for every pair of left-invariant means μ and $\mu', \mu(f) = \mu'(f)$. If G is the additive group of integers, then (i) f is naturally integrable if, and only if,

$$\lim n^{-1}\Sigma f(j+i)(1 \leqq i \leqq n)$$

exists uniformly in j, and (ii) the associated natural measure ν is convex; that is, for every pair of naturally measurable sets of integers E_0 and E_1 with $E_0 \subset E_1$, there is a monotone family of naturally measurable sets $E_t(0 \leqq t \leqq 1)$ such that $\nu(E_t)(0 \leqq t \leqq 1)$ is a closed interval. Analogous results hold for the presently known amenable groups.

An order-preserving linear functional μ — or integral — defined on the space $B(G)$ of bounded, real-valued functions defined on a group G, is a *(left)-invariant mean* if $\mu(c) = c$ for all constants c and if

$$\mu f = \int f(xy)d\mu(y) \tag{1}$$

for all $x \in G$; and G is *amenable* if such a μ exists.

If $\mu f = \mu' f$ for all invariant means μ and μ', then f is *(left) naturally integrable*. This section is concerned with characterizing the set, $\mathscr{N}$, of left naturally integrable functions. As a preliminary, a necessary and sufficient condition for G to be amenable will be given.

For each finite subset α of G and each $f \in B(G)$, the convolution of α with f, $\alpha * f$ is defined by

$$(\alpha * f)(y) = \frac{1}{|\alpha|} \sum f(xy)(x \in \alpha) \tag{2}$$

where $|\alpha|$ is the cardinality of α. Plainly, for any invariant μ,

$$\mu f \leqq \overline{\alpha * f}\,, \quad \text{for each} \quad \alpha\,, \tag{3}$$

where $\bar{f}$ is the supremum of f.

Summarizing,

$$\mu f \leqq p(f) \tag{4}$$

where

$$p(f) = \inf_{\alpha} \overline{\alpha * f}\,. \tag{5}$$