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ON THE EXISTENCE OF CAPILLARY FREE SURFACES IN THE ABSENCE OF GRAVITY

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If we were to put water into a glass cylindrical container of circular cross section and transport it to outer space, the surface of the water would be uniquely determined as a lower hemisphere. Concus and Finn [1] have shown that if the circular cross section is replaced by a square section then no such surface can exist as a graph of a function. The question then arises, for what kind of cross section can we expect the existence of a surface in the form of a graph?

Mathematically, this question can be formulated as follows: Let Ω be the cross section of the cylinder and u be a capillary free surface defined over Ω. By the least action principle of physics, u would minimize the energy functional

$$E[u] = \sigma \iint_{\Omega} \sqrt{1 + |\nabla u|^2}\, dxdy - \sigma\lambda \int_{\Sigma} u\, ds \tag{1.1}$$

subject to the volume constraint

$$\iint_{\Omega} u\, dxdy = \text{constant} \tag{1.2}$$

where Σ is the boundary of Ω, ds is the arc length measure on Σ and ∇u is the gradient of u. The physical interpretation of (1.1) is as follows: The first term gives the potential energy in the free surfaces; the constant σ is referred to as the surface tension. The second term gives the wetting energy due to the boundary adhesion. The dimensionless constant λ satisfies $|\lambda| \leqq 1$ and depends on the material of the wall and the fluid. For glass and water, λ is close to 1, and for glass and mercury, λ is negative. We mention that the case $\lambda > 1$ corresponds physically to a situation in which adhesion dominate, so that the fluid would spread out along the walls and no equilibrium surface would exist. This phenomenon is observed, e.g., with liquid helium and glass.

The equilibrium condition $\delta E[u] = 0$ under the constraint (1.2) is expressed by the Euler equations:

$$\operatorname{div} Tu = H \quad \text{in} \quad \Omega \tag{1.3}$$

$$Tu \cdot \nu = \lambda \quad \text{on} \quad \Sigma \tag{1.4}$$

where $Tu\langle u_x/\sqrt{1 + |\nabla u|^2}, u_y/\sqrt{1 + |\nabla u|^2}\rangle$, H is a constant which is