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CANCELLATION OF MODULES AND GROUPS AND STABLE RANGE OF ENDOMORPHISM RINGS

R. B. WARFIELD, JR.

In this paper, various cancellation properties for a module are shown to follow from facts about the stable range of its endomorphism ring, and a number of applications to the structure of modules, Abelian groups, and complex tori are given.

In the first section we introduce a property which we call the *n-substitution* property for a module, which holds if and only if the integer n is in the stable range of its endomorphism ring. Direct sums and summands of modules with this property are studied, and a slight generalization of Vasershtein's theorem on the stable range of matrix rings is obtained. In the second section two strong cancellation properties for a module are shown to be equivalent to one being in the stable range of its endomorphism ring, and an application is obtained concerning modules whose endomorphism ring modulo its radical is von Neumann regular. Bass showed that if S is a commutative J-Noetherian ring of J-dimension d, and R is a finite S-algebra, then $d+1$ is in the stable range for R. If S is not Noetherian, this is not a strong enough result to apply to endomorphism rings, and it is improved in the third section to say that the endomorphism ring of any finitely presented module over such a ring has $d+1$ in the stable range. In section four are two stronger results, one concerning finitely presented modules over rings of Krull dimension one in the classical sense (in terms of chains of prime ideals) and the other concerning projective modules over rings having Krull dimension one in the noncommutative sense. The final section contains several applications to torsion-free Abelian groups, modules over valuation rings, and complex tori. It is shown that a torsion-free algebra of finite rank over a semilocal principal ideal domain or over a (possibly non-Noetherian) valuation ring has one in the stable range. This implies cancellation theorems for torsion-free Abelian groups of finite rank satisfying certain divisibility conditions and also for torsion-free modules of finite rank over a valuation ring. It is shown that two is in the stable range for any torsion-free algebra of finite rank over a principal ideal domain. As an application, it is shown that two torsion-free Abelian groups A and B of finite rank are of the same genus if and only if for some positive integer n, $A^n \cong B^n$, so that, in particular, if A, B, and C are torsion-free groups of finite rank and $A \oplus B \cong A \oplus C$, then for some positive integer n, $B^n \cong C^n$. There are corresponding ap-